

# A non-trivial upper bound on the threshold bias of the Oriented-cycle game

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## Abstract

In the Oriented-cycle game, introduced by Bollobás and Szabó, two players, called OMaker and OBreaker, alternately direct edges of  $K_n$ . OMaker directs exactly one edge, whereas OBreaker is allowed to direct between one and  $b$  edges. OMaker wins if the final tournament contains a directed cycle, otherwise OBreaker wins. Bollobás and Szabó conjectured that for a bias as large as  $n - 3$  OMaker has a winning strategy if OBreaker must take exactly  $b$  edges in each round. It was shown recently by Ben-Eliezer, Krivelevich and Sudakov, that OMaker has a winning strategy for this game whenever  $b \leq n/2 - 2$ . In this paper, we show that OBreaker has a winning strategy whenever  $b \geq 5n/6 + 2$ . Moreover, in case OBreaker is required to direct exactly  $b$  edges in each move, we show that OBreaker wins for  $b \geq 19n/20$ , provided that  $n$  is large enough. This refutes the conjecture by Bollobás and Szabó.

## 1 Introduction

We consider biased orientation games, as discussed by Ben-Eliezer, Krivelevich and Sudakov in [6]. In orientation games, the board consists of the edges of the complete graph  $K_n$ . In the  $(a : b)$  orientation game, the two players called OMaker and OBreaker, direct previously undirected edges alternately. OMaker starts, and in each round, OMaker directs between one and  $a$  edges, and then OBreaker directs between one and  $b$  edges. At the end of the game, the final graph is a tournament on  $n$  vertices. OMaker wins the game if this tournament has some predefined property  $\mathcal{P}$ . Otherwise, OBreaker wins.

Orientation games can be seen as a modification of  $(a : b)$  Maker-Breaker games, played on the complete graph  $K_n$ . The game is played by two players, Maker and Breaker, who alternately claim  $a$  and  $b$  edges, respectively. Maker wins if the subgraph consisting of her edges satisfies some given monotone-increasing property  $\mathcal{P}$ . Otherwise, Breaker wins. Maker-Breaker games have been widely studied (cf. [1], [2], [3], [5], [8], [10], [11], [14]), and it is quite natural to translate typical questions about Maker-Breaker games to orientation games.

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For instance, Beck [4] studied the so-called *Clique game*, proving that in the  $(1 : 1)$  game, the largest clique that Maker is able to build is of size  $(2 - o(1)) \log_2(n)$ . Motivated by this result, an orientation game version of the Clique game was considered in [9]: Given a tournament  $T_k$  on  $k$  vertices, it was proven there that for  $k \leq (2 - o(1)) \log_2(n)$  OMaker can ensure that  $T_k$  appears in the final tournament (independent of the choice of  $T_k$ ), while for  $k \geq (4 + o(1)) \log_2(n)$  OBreaker always can prevent a copy of the predefined tournament  $T_k$ .

In this work we only consider orientation games with  $a = 1$ . We refer to the  $(1 : 1)$  orientation game as the *unbiased* orientation game, and the  $(1 : b)$  orientation game as the *b-biased orientation game* when  $b > 1$ . Increasing  $b$  can only help OBreaker, so the game is *bias monotone*. Therefore, any such game (besides degenerate games where  $\mathcal{P}$  is a property that is satisfied by every or by no tournament on  $n$  vertices) has a *threshold*  $t(n, \mathcal{P})$  such that OMaker wins the  $b$ -biased game when  $b \leq t(n, \mathcal{P})$  and OBreaker wins the game when  $b > t(n, \mathcal{P})$ .

In a variant, OBreaker is required to direct exactly  $b$  edges in each round. We refer to this variant as the *strict b-biased orientation game*, where *the strict rules apply*. Accordingly, we say the *monotone rules apply* in the game we defined above - when OBreaker is free to direct between one and  $b$  edges. Playing the exact bias in every round may be disadvantageous for OBreaker, so the existence of a threshold as for the monotone rules is unclear in general. We therefore define  $t^+(n, \mathcal{P})$  to be the largest value  $b$  such that OMaker has a strategy to win the strict  $b$ -biased orientation game, and  $t^-(n, \mathcal{P})$  to be the largest integer such that for every  $b \leq t^-(n, \mathcal{P})$ , OMaker has a strategy to win the strict  $b$ -biased orientation game. (The definition of these different threshold functions is motivated by the study of Avoider-Enforcer games, cf. [13], [12].) Trivially,  $t(n, \mathcal{P}) \leq t^-(n, \mathcal{P}) \leq t^+(n, \mathcal{P})$  holds.

The threshold bias  $t(n, \mathcal{P})$  was investigated by Ben-Eliezer, Krivelevich and Sudakov [6] for several orientation games. They showed for example that  $t(n, \mathcal{H}) = (1 + o(1))n/\ln n$ , where  $\mathcal{H}$  is the property to contain a directed Hamilton cycle. However, the relation between all three parameters in question is still widely open. It is not even clear whether  $t^-(n, \mathcal{H})$  and  $t^+(n, \mathcal{H})$  need to be distinct values.

In this work, we study the *Oriented-cycle game*, introduced by Bollobás and Szabó [7], which is an orientation game where  $\mathcal{P} = \mathcal{C}$  is the property of containing a directed cycle. That is, OMaker wins if the final tournament contains a directed cycle, and OBreaker wins if the final tournament is transitive. The Maker-Breaker variant of this game was studied in [5], where it is shown that Maker has a strategy to claim a cycle in the  $(1 : b)$  game if and only if  $b < \lceil n/2 \rceil - 1$ .

For an upper bound on the threshold bias in the orientation game, Bollobás and Szabó observed that  $t^+(n, \mathcal{C}) \leq n - 3$ . Indeed, with a short case distinction, it can be verified that for  $b \geq n - 2$ , OBreaker can always ensure that immediately after each round there exists a subset  $\{v_1, \dots, v_k, v_{k+1}\} \subseteq V = V(K_n)$  such that for every  $1 \leq i \leq k$  and  $v \in V \setminus \{v_1, \dots, v_i\}$  the edge  $v_i v$  is directed from  $v_i$  to  $v$ ; and every directed edge in  $V \setminus \{v_1, \dots, v_k\}$  starts in  $v_{k+1}$ . If these properties hold, there is neither a directed cycle nor an edge that could close such a cycle, i.e. OBreaker wins. We refer to this strategy as the *trivial strategy*.

The strict version of the Oriented-cycle game was studied by Alon (unpublished, cf. [7]), and later by Bollobás and Szabó [7]. They show that  $t^+(n, \mathcal{C}) \geq \lfloor (2 - \sqrt{3})n \rfloor$ . Moreover, they remark that the proof also works for the monotone rules, implying that  $t(n, \mathcal{C}) \geq \lfloor (2 - \sqrt{3})n \rfloor$ . Finally, they conjectured that  $t^+(n, \mathcal{C}) = n - 3$ . In [6], Ben-Eliezer, Krivelevich and Sudakov

improve the lower bound and show that for  $b \leq n/2 - 2$ , OMaker has a strategy guaranteeing a cycle in the  $b$ -biased orientation game, i.e.  $t(n, \mathcal{C}) \geq n/2 - 2$ . In the main result of this paper we refute the conjecture of Bollobás and Szabó and give a strategy for OBreaker to prevent cycles when  $b \geq 19n/20$  for large  $n$ .

**Theorem 1.1.** *For large enough  $n$ ,  $t^+(n, \mathcal{C}) \leq 19n/20 - 1$ .*

In the monotone game our strategy simplifies and gives a winning strategy already when  $b = 5n/6 + 2$ .

**Theorem 1.2.**  $t(n, \mathcal{C}) \leq 5n/6 + 1$ .

The remaining part of the paper is organized as follows. First we introduce some general notation and terminology. In Section 2, we introduce some necessary concepts and prove Theorem 1.2. In Section 3, we describe a strategy for OBreaker in the strict game, we prove that this strategy constitutes a winning strategy in Section 4 and 5. We finish the paper with some concluding remarks.

## 1.1 General notation and terminology

Let  $V = [n]$  and let  $D \subseteq V \times V$  be a digraph. We call elements  $(v, w) \in D$  *arcs* and the underlying set  $\{v, w\}$  a *pair* or an *edge*. An arc  $(v, v)$  is called a *loop* and  $(v, w)$  is called the *reverse arc* for  $(w, v)$ . In this work we are only concerned with *simple* digraphs, without loops and reverse arcs. For an arc  $e \in D$ , we write  $e^+$  for its tail and  $e^-$  for its head, i.e.  $e = (e^+, e^-)$ . By  $\overleftarrow{e}$  we denote the reverse arc of  $e$ . For a subdigraph  $S \subseteq D$ , we denote by  $S^+$  the set of all tails  $e^+$  for  $e \in S$ , and by  $S^-$  the set of all heads  $e^-$  for  $e \in S$ . It is convenient to denote by  $\overleftarrow{D}$  the set of all reverse arcs of  $D$ , which we call the *dual* of  $D$ , that is  $\overleftarrow{D} := \{\overleftarrow{e} : e \in D\}$ . Moreover, the set  $\mathcal{A}(D) := (V \times V) \setminus (D \cup \overleftarrow{D} \cup \mathcal{L})$  denotes the set of all *available arcs*, where  $\mathcal{L} = \{(v, v) : v \in V\}$  is the set of all loops. Note that  $\mathcal{A}(D)$  is symmetric, i.e. if  $(v, w) \in \mathcal{A}(D)$  then also  $(w, v) \in \mathcal{A}(D)$ . We generalize the notation of an arc and say the  $k$ -tuple  $(v_1, \dots, v_k)$  *induces a transitive tournament in  $D$* , if for all  $1 \leq i < j \leq k$  we have that  $(v_i, v_j) \in D$ . For two disjoint sets  $A, B \subseteq V$  we call the pair  $(A, B)$  a *uniformly directed biclique*, or short *UDB*, if for all  $v \in A, w \in B$  we have that  $(v, w) \in D$ . We say the sequence  $P = (e_1, \dots, e_k)$  is a *directed path* (or simply a *path*) in  $D$  if all  $e_i \in D$  and for all  $1 \leq i < k$  we have that  $e_i^- = e_{i+1}^+$ . In this case we say that  $P$  is a  $e_1^+ - e_k^-$ -path. We also write  $P = v_1, \dots, v_\ell$  to denote the path  $P = (e_1, \dots, e_{\ell-1})$  where  $e_i = (v_i, v_{i+1})$ .

In our proofs we are concerned how  $D$  behaves on certain subsets of the vertices. For a subset  $A \subseteq V$  we denote by  $D(A)$  the directed subgraph of  $D$  of arcs spanned by  $A$ . Formally,  $D(A) := D \cap (A \times A)$ . For two (not necessarily disjoint) sets  $A, B \subseteq V$ , we set  $D(A, B) := D \cap (A \times B)$  to be the set of those edges in  $D$  that start in  $A$  and end in  $B$ . To shorten the notation, we also set  $D(v, B) := D(\{v\}, B)$  and  $D(A, v) := D(A, \{v\})$  for  $v \in V$ . Moreover, to describe the sizes of these edge sets, we let  $e_D(A) := |D(A)|$ ,  $e_D(A, B) := |D(A, B)|$ ,  $e_D(v, B) := |D(v, B)|$  and  $e_D(A, v) := |D(A, v)|$ . Given the digraph  $D$ , we also might want to delete or add some edge  $e$ . To simplify the notation, we write  $D + e := D \cup \{e\}$  and  $D - e := D \setminus \{e\}$ .

Recall that the Oriented-cycle game is played on the edge set of  $K_n$  where we assume that

$V = V(K_n) = [n]$ . We say a player *directs the edge*  $(v, w)$  if  $\mathfrak{o}^1$  directs the pair  $\{v, w\}$  from  $v$  to  $w$ . That is, the player chooses the arc  $(v, w)$  to belong to the final digraph, and dismisses the arc  $(w, v)$  from the board. After some round  $r$ , we shall refer to  $D \subseteq V \times V$  as the sub-digraph of already directed edges (arcs) by either player. We say a player *closes a cycle in  $D$  by directing some edge*  $(v, w)$  if there exists a  $w$ - $v$ -path in  $D$ . Note that if a player can close a cycle in  $D$ , then  $\mathfrak{o}$  can close a triangle (consider the shortest cycle a player can close, and consider any cord).

## 2 The Oriented-cycle game – monotone rules

There are two essential concepts to our proof, the aforementioned *UDB*'s and  $\alpha$ -structures. A *UDB* is a complete bipartite digraph where all the edges are directed in the same direction (i.e. from  $A$  to  $B$ ). Our goal is to create a *UDB*  $(A, B)$  such that both parts fulfil  $|A|, |B| \leq b$  and  $A \cup B = V$ . Suppose the following situation would be given to us for free. There is a partition  $A \cup B = V$  such that the pair  $(A, B)$  forms a *UDB* in  $D$ , both parts fulfil  $|A|, |B| \leq b$  and both sets  $A$  and  $B$  are empty (i.e.  $D(A) = D(B) = \emptyset$ ). OBreaker could then follow the trivial strategy inside  $A$  and  $B$  respectively (as OBreaker wins on  $K_{b+2}$ ), even when the strict rules apply.

However, while building such a *UDB*, OMaker can direct edges inside these sets, and OBreaker needs to control those. Moreover, to optimise the bias, OBreaker should be able to control those edges inside  $A$  and  $B$  with as few edges as possible. To handle this obstacle, we introduce certain structures which we call  $\alpha$ -structures and a way to incorporate new (i.e. OMaker's) edges into an existing  $\alpha$ -structure. Before we move on to study these special structures let us mention that the idea of building a big *UDB* quickly comes up again in the proof of Theorem 1.1 in the subsequent sections. However, the requirement of directing exactly  $b$  edges in each move puts some serious restrictions on the power of  $\alpha$ -structures. So for the strict rules, we then consider only special  $\alpha$ -structures, that are more robust to adding more edges.

### 2.1 $\alpha$ -structures

The definition of an  $\alpha$ -structure looks quite technical at first sight. So let us motivate the idea behind it.

Suppose OMaker's strategy is to build a long path first. (This indeed is the strategy for OMaker in the so far best-known lower-bound proof in [6].) Let  $P = (e_1, \dots, e_k)$  be a directed path of length  $k$  in  $D$  with arcs  $e_i = (v_i, v_{i+1})$ , and suppose OMaker enlengthens  $P$  by directing an edge  $(v_{k+1}, w)$  for some  $w \in V$ . Then all the pairs  $\{w, v_i\}$  for  $1 \leq i \leq k$  constitute potential threats as directing any  $(w, v_i)$  would close a cycle (we call such pairs *immediate threats*). So OBreaker better directs all edges  $(v_i, w)$  in his next move, we say OBreaker *closes* immediate threats. This way, OBreaker fills up the missing arcs of an evolving transitive tournament with *spine*  $e_1, \dots, e_k$ . Then formally, OBreaker sets  $v_{k+2} := w$ ,  $e_{k+1} := (v_{k+1}, w)$  and directs all edges  $(e_i^+, w)$  for  $i \leq k$ . Clearly, as long as there are isolated vertices, OMaker could follow

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<sup>1</sup>We use the Hungarian 3<sup>rd</sup> person pronoun  $\mathfrak{o}$  whenever an appearance of {he, she} is required (and  $\mathfrak{o}$ 's for {his, her}). Please replace accordingly, either to your liking or at random.

this strategy and increase the number of threats that OBreaker *has to close immediately* by one in each move.

A naïve strategy for OBreaker would be to close immediate threats only. However, then OMaker could claim two vertex-disjoint paths of linear length, say  $P_1 = v_1, \dots, v_{\varepsilon n}$  and  $P_2 = w_1, \dots, w_{\varepsilon n}$ , for some  $\varepsilon > 0$ , and OBreaker would not claim edges between these two paths. Directing the arc  $(v_{\varepsilon n}, w_1)$  then, OMaker could suddenly create  $(\varepsilon n)^2 > b$  immediate threats, which OBreaker cannot close in the next move.

By defining the  $\alpha$ -structure we prevent such a situation. Moreover, we show that “building a long path” is the best possible strategy for OMaker in the following sense: No matter how OMaker plays, OBreaker has a strategy such that in round  $r$ ,  $\mathfrak{O}$  has to direct at most  $r$  edges to close immediate threats.

**Definition 2.1.** Let  $V$  be a set of vertices, and let  $D \subset (V \times V) \setminus \mathcal{L}$  be a digraph without loops and reverse arcs. Then  $D$  is called an  $\alpha$ -structure of rank  $r$  if there exist  $k \leq r$  arcs  $e_1, \dots, e_k \in D$  such that the map

$$\alpha : \{(i, j) : 1 \leq i \leq j \leq k\} \rightarrow D$$

defined by  $\alpha((i, j)) := (e_i^+, e_j^-)$ , is a surjection. The arcs  $e_1, \dots, e_k$  are called decisive arcs of the  $\alpha$ -structure  $D$ .

In Figure 1, we give three simple examples of  $\alpha$ -structures. In our strategy that we describe later, the arcs  $e_1, \dots, e_k$  are edges directed by OMaker (though not necessarily in that order). Note that the arcs  $e_1, \dots, e_k$  uniquely determine the  $\alpha$ -structure  $D$ . In particular,  $D^+ = \{e_1^+, \dots, e_k^+\}$  and  $D^- = \{e_1^-, \dots, e_k^-\}$ .

The condition  $k \leq r$  in the definition above might seem somewhat artificial at first sight. All the properties about  $\alpha$ -structures in this subsection are still true (or slight variations of them) if we require  $k = r$  in the definition. However, in the next subsection, this relaxed definition makes it easier to handle OBreaker’s strategy.

Now, let us capture some immediate facts about  $\alpha$ -structures. The first proposition states that an  $\alpha$ -structure is *self-dual* in the following sense.

**Proposition 2.2.**  $D$  is an  $\alpha$ -structure of rank  $r$  if and only if  $\overleftarrow{D}$  is an  $\alpha$ -structure of rank  $r$ .

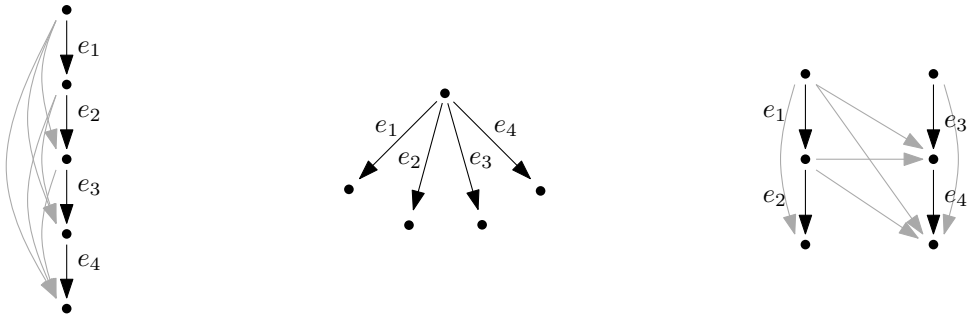


Figure 1: Three  $\alpha$ -structures of rank four that are not  $\alpha$ -structures of rank three.

*Proof.* If  $e_1, \dots, e_k$  are decisive arcs in  $D$ , then  $\overleftarrow{e_k}, \dots, \overleftarrow{e_1}$  are decisive arcs in  $\overleftarrow{D}$ .  $\square$

The next lemma says that restricting to induced subgraphs does not destroy  $\alpha$ -structures.

**Lemma 2.3.** *Let  $D$  be an  $\alpha$ -structure of rank  $r$  on vertex set  $V$ . Then for any subset  $V' \subseteq V$ , we have that  $D(V')$  is an  $\alpha$ -structure of rank  $r$ .*

*Proof.* Let  $v \in V$ . It is enough to prove the statement for  $V' = V \setminus \{v\}$ . Let  $S_0 = \{e_1, \dots, e_k\}$  be a set of decisive arcs for  $D$ , with  $k \leq r$ , as given by the Definition 2.1. In the following we construct a set  $S_k$  which then turns out to be a set of decisive arcs of  $D' := D(V')$ . For  $i = 1, \dots, k$  do the following iteratively:

1. If  $v = e_i^+$  and there is an arc  $g \in D' \setminus S_{i-1}$  with  $g^- = e_i^-$  such that  $g$  is forced by  $e_i$  (there exists  $s < i$  such that  $g^+ = e_s^+$ ), then set  $f_i := g$  and  $S_i := S_{i-1} - e_i + f_i$ .
2. If  $v = e_i^-$  and there is an arc  $g \in D' \setminus S_{i-1}$  with  $g^+ = e_i^+$  such that  $g$  is forced by  $e_i$  (there exists  $j > i$  such that  $g^- = e_j^-$ ), then set  $f_i := g$  and  $S_i := S_{i-1} - e_i + f_i$ .
3. If  $v \notin \{e_i^+, e_i^-\}$ , then set  $f_i := e_i$  and  $S_i := S_{i-1} - e_i + f_i$ .
4. In any other case set  $S_i := S_{i-1} - e_i$ .

**Claim 2.4.** *Let  $I \subseteq [k]$  be the index set of those arcs  $e_i$  removed in Case 4, and let  $M := \{(i, j) : 1 \leq i \leq j \leq k, i, j \notin I\}$ . Then  $\alpha'((i, j)) := (f_i^+, f_j^-)$  defines a surjective map from  $M$  to  $D'$ . So  $D'$  is an  $\alpha$ -structure with decisive arcs  $S_k$ .*

To show that  $\alpha'(M) \subseteq D'$  let  $(i, j) \in M$  and assume that  $\alpha'((i, j)) = (f_i^+, f_j^-) \notin D'$ . Then either  $f_i^+ = e_i^+$  (Case 2 and 3) or  $f_i^+ = e_s^+$  for some  $s < i$  (Case 1). Analogously,  $f_j^- = e_t^-$  for some  $t \geq j$ . Hence  $(f_i^+, f_j^-) = (e_s^+, e_t^-) = \alpha((s, t)) \in D$  for some  $s \leq t$ . By definition of  $f_i$  and  $f_j$ ,  $f_i^+ \neq v \neq f_j^-$ , hence  $(f_i^+, f_j^-) \in D'$ , a contradiction.

Let now  $f \in D'$ . If  $f = f_i$  for some  $i \notin I$ , then  $f = \alpha'((i, i))$ . So, assume that  $f \notin S_k$ . Note that this implies  $f \notin S_i$  for  $i \leq k$ . Since  $D$  is an  $\alpha$ -structure,  $f = (e_i^+, e_j^-)$  for some  $1 \leq i \leq j \leq k$ . If  $e_i^- = v$ , then Case 2 applies since  $f \notin S_{i-1}$  is such an arc forced by  $e_i$  (though the algorithm chooses  $g$  different from  $f$ ). Then  $e_i^+ = g^+ = f_i^+$  and  $i \notin I$ . If  $e_i^- \neq v$ , then  $e_i = f_i$  (Case 3) and  $i \notin I$ . Analogously,  $e_j^- = f_j^-$  and  $j \notin I$ . Therefore,  $f = (f_i^+, f_j^-) = \alpha'((i, j))$ , which finishes the proof of the claim and the lemma.  $\square$

Before we prove that  $\alpha$ -structures are indeed acyclic, let us look at paths in them. The next lemma roughly says that any path in an  $\alpha$ -structure can be controlled through its decisive arcs. An illustration can be found in Figure 2.

**Proposition 2.5.** *Let  $D$  be an  $\alpha$ -structure of rank  $r$  with decisive arcs  $e_1, \dots, e_k$  and  $k \leq r$ . Let  $P = (f_1, \dots, f_t)$  be a path in  $D$ . Then there are decisive arcs  $e_{i_1}, e_{j_1}, e_{i_2}, e_{j_2}, \dots, e_{i_t}, e_{j_t}$  (not necessarily distinct) such that  $f_s = (e_{i_s}^+, e_{j_s}^-)$  and  $i_1 \leq j_1 \leq i_2 \leq j_2 \leq \dots \leq i_t \leq j_t$ .*

*Proof.* Since the function  $\alpha((i, j)) := (e_i^+, e_j^-)$  maps  $\{(i, j) : 1 \leq i \leq j \leq k\}$  surjectively onto  $D$ , the existence of the decisive edges  $e_{i_s}, e_{j_s}$  with  $i_s \leq j_s$  such that  $f_s = (e_{i_s}^+, e_{j_s}^-)$  follows.

Now, suppose there was some  $s \in [t-1]$  such that  $j_s > i_{s+1}$ . Then the arc  $(f_{s+1}^+, f_s^-) = (e_{i_{s+1}}^+, e_{j_s}^-) = \alpha((i_{s+1}, j_s))$  would be a loop in  $D$ , a contradiction.  $\square$

The following property is crucial for our orientation game. Recall that  $\mathcal{A}(D) = (V \times V) \setminus (D \cup \overleftarrow{D} \cup \mathcal{L})$  denotes the set of available arcs.

**Proposition 2.6.** *If  $D$  is an  $\alpha$ -structure, then for every available  $e \in \mathcal{A}(D)$  we have that  $D + e$  is acyclic.*

*Proof.* Suppose there was a path  $P = (f_1, \dots, f_t)$  in  $D$  and an edge  $e \in D \cup \mathcal{A}(D)$  such that  $P + e$  forms a directed cycle. Let  $e_{i_1}, e_{j_1}, e_{i_2}, e_{j_2}, \dots, e_{i_t}, e_{j_t}$  be given by Proposition 2.5. Then since  $i_1 \leq j_t$  we have that  $(f_1^+, f_t^-) = (e_{i_1}^+, e_{j_t}^-) = \alpha((i_1, j_t)) \in D$ . So,  $e = (f_t^-, f_1^+) \in \overleftarrow{D}$  and hence not available, a contradiction.  $\square$

In the light of our orientation game, we pin down the following important implication.

**Corollary 2.7.** *For some subset  $V' \subseteq V$ , suppose that in the Oriented-cycle game, OBreaker maintains that  $D(V')$  is an  $\alpha$ -structure (of some rank  $r$ ). Then there is no cycle in  $D(V')$  and OMaker cannot close a cycle inside  $V'$  in  $\mathfrak{o}$ 's next move.*

In order for OBreaker to maintain an  $\alpha$ -structure on some subset of the vertices we need to know how to incorporate OMaker's edge into such a structure. The following is one of the key lemmas in OBreaker's strategy.

**Lemma 2.8.** *Let  $D$  be an  $\alpha$ -structure of rank  $r$  on vertex set  $V$ , and let  $e \in \mathcal{A}(D)$  be an available arc. Then there exist at most  $\min\{r, |V| - 2\}$  available arcs  $\{f_1, \dots, f_t\} \subseteq \mathcal{A}(D)$  such that  $D' = D \cup \{e, f_1, \dots, f_t\}$  is an  $\alpha$ -structure of rank  $r+1$ . Moreover,  $D'^+ = D^+ \cup \{e^+\}$  and  $D'^- = D^- \cup \{e^-\}$ .*

By adding  $e$  to the  $\alpha$ -structure  $D$  we mean a strategy for OBreaker to direct the arcs  $\{f_1, \dots, f_t\}$  given by the previous lemma. Before we prove the lemma, we need one more definition. Let  $D$  be an  $\alpha$ -structure with decisive arcs  $S = \{e_1, \dots, e_k\}$ , and let  $x \in V$ . We set

$$\begin{aligned} In(x) &:= \left\{ e_i : \text{there exists a path } P = (e_i, e_{j_1}, \dots, e_{j_m}) \text{ s.t. } x = e_{j_m}^- \right\}, \\ Out(x) &:= \left\{ e_i : \text{there exists a path } P = (e_{j_1}, \dots, e_{j_m}, e_i) \text{ s.t. } x = e_{j_1}^+ \right\}. \end{aligned}$$

The following proposition is rather simple.

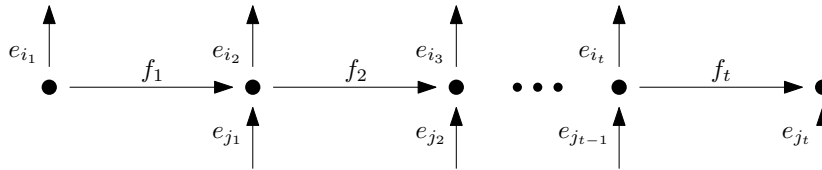


Figure 2: Illustrating Proposition 2.5.

**Proposition 2.9.** *Let  $D$  be an  $\alpha$ -structure with decisive arcs  $e_1, \dots, e_k$ . Further, let  $x, y \in V$  be vertices such that  $(x, y) \in \mathcal{A}(D)$ . Then for all  $e_i \in \text{In}(x)$ ,  $e_j \in \text{Out}(y)$ :  $i < j$ . In particular,  $\text{In}(x) \cap \text{Out}(y) = \emptyset$ .*

*Proof.* Let  $e_i \in \text{In}(x)$ ,  $e_j \in \text{Out}(y)$  and let  $P_i$  be the corresponding  $e_i^+$ - $x$ -path starting with  $e_i$ , and let  $P_j$  be the corresponding  $y$ - $e_j^-$ -path ending with  $e_j$ . Assume  $j \leq i$ , then since  $\alpha$  is a surjection, the arc  $(e_j^+, e_i^-) = \alpha((j, i)) \in D$ . But then the concatenation of  $(P_j - e_j)$ ,  $(e_j^+, e_i^-)$  and  $(P_i - e_i)$  is a directed walk in  $D$  from  $y$  to  $x$ , i.e. contains a directed path from  $y$  to  $x$ . By Proposition 2.6,  $(x, y)$  is not available, a contradiction.  $\square$

We are now ready to prove the above lemma.

*Proof of Lemma 2.8.* Let  $D$  be the  $\alpha$ -structure of rank  $r$  on a vertex set  $V$ , let  $S = \{e_1, \dots, e_k\}$  be a set of decisive arcs with  $k \leq r$ , and let  $e = (v, w) \in \mathcal{A}(D)$  be an available arc. Set  $\ell := \min\{i : e_i \in \text{Out}(w)\}$  if  $\text{Out}(w) \neq \emptyset$ , and  $\ell := k + 1$  otherwise. For all  $i < \ell$ , set  $f_i := (e_i^+, w)$ , and for all  $i \geq \ell$ , set  $f_i := (v, e_i^-)$ . We claim that for all  $1 \leq i \leq k$ , either  $f_i \in D$  or  $f_i \in \mathcal{A}(D)$ .

First, let  $i < \ell$  and suppose for a contradiction that  $f := (w, e_i^+) \in D$ . Since  $\alpha$  is a surjection onto  $D$ ,  $f = (e_{j_1}^+, e_{j_2}^-)$  for some  $j_1 \leq j_2$ . Now,  $j_2 < i$  since otherwise  $(e_i^+, e_{j_2}^-) = \alpha((i, j_2))$  is a loop in  $D$ . Furthermore, since  $e_{j_1}^+ = w$ , by definition  $e_{j_1} \in \text{Out}(w)$ , so  $\ell \leq j_1$ . But this implies  $\ell < i$ , a contradiction.

Now let  $i \geq \ell$ . The only additional observation we need to make here is that by Proposition 2.9,  $\ell > j$  for all  $e_j \in \text{In}(v)$ . The rest is completely analogous to the first case. So we can assume that either  $(v, e_i^-) \in \mathcal{A}(D)$ , or  $(v, e_i^-) \in D$ .

We now check that  $D'$  is an  $\alpha$ -structure of rank  $r + 1$ , where  $S' = \{e'_1, \dots, e'_{k+1}\}$  with

$$e'_i = \begin{cases} e_i & \text{if } i < \ell \\ e & \text{if } i = \ell \\ e_{i-1} & \text{if } i > \ell \end{cases}$$

is a set of decisive arcs. That is, for  $M := \{(i, j) : 1 \leq i \leq j \leq k + 1\}$  we need to show that  $\alpha'((i, j)) = (e_i'^+, e_j'^-)$  defines a surjection from  $M$  to  $D'$ , i.e.  $\alpha'(M) = D'$ .

Let  $M' = \{(i, j) : 1 \leq i \leq j \leq k + 1, i, j \neq \ell\}$ . Then  $\alpha'(M') = \text{Im}(\alpha) = D$ . Moreover, for  $i < \ell$  we have  $\alpha((i, \ell)) = (e_i'^+, e_\ell'^-) = (e_i^+, w) = f_i$ , and  $\alpha((\ell, \ell)) = e$ , and for  $\ell + 1 \leq i \leq k + 1$  we have  $\alpha((\ell, i)) = (e_\ell'^+, e_i'^-) = (v, e_{i-1}^-) = f_{i-1}$ . Thus,  $\alpha'(M \setminus M') = \{e, f_1, \dots, f_t\}$ .

Finally, for every existing arc  $e_i \in S$ , we added at most one new arc  $f_i$ . But also, for every vertex  $z \in V \setminus \{v, w\}$  at most one of the  $f_i$  contains  $z$ . So  $|\{f_1, \dots, f_k\} \cap \mathcal{A}(D + e)| \leq \min\{k, |V| - 2\} \leq \min\{r, |V| - 2\}$ .  $\square$

## 2.2 OBreaker's strategy for the monotone rules

*Proof of Theorem 1.2.* Recall that OMaker and OBreaker alternately direct edges of  $K_n$ , where OMaker directs exactly one edge in each round, and OBreaker directs at least one and at most  $b$  edges in each round, where  $b \geq 5n/6 + 2$ . OMaker's goal is to close a directed cycle,



whereas OBreaker's goal is to prevent this. First, we provide OBreaker with a strategy, then we prove that  $\mathfrak{o}$  can follow that strategy and that it constitutes a winning strategy. At any point during the game let  $D$  denote the digraph of already chosen arcs. By the rules of the game,  $D$  has no loops and no reverse arcs.

The main idea of OBreaker's strategy is to maintain that  $D$  consists of a *UDB*  $(A, B)$  and two  $\alpha$ -structures, located in  $V \setminus B$  and  $V \setminus A$  respectively, in such a way that each arc of  $D$  starts in  $A$  or ends in  $B$ . For an illustration, see Figure 3. We show shortly that this is enough to prevent cycles throughout the game. To succeed with a bias  $b \geq 5n/6 + 2$ , we also need to keep control on the size of the UDB and the rank of the mentioned  $\alpha$ -structures. For that reason, we divide the strategy into three stages, in each, we maintain different size conditions. In particular, throughout Stage I, OBreaker builds up two large "buffer sets"  $A' \subseteq A$  and  $B' \subseteq B$ , until their size is  $n - b$ . These buffer sets then allow OBreaker to play on two - not necessarily disjoint - boards in Stage II and III, each board having at most  $b$  vertices.

If the strategy asks OBreaker to direct an arc  $(x, y)$  where  $(x, y) \in D$  already, then OBreaker ignores that command and proceeds to the next one. If the strategy asks OBreaker to direct an arc  $(x, y)$  where  $(y, x) \in D$  already, then OBreaker forfeits the game.

In **Stage I**, OBreaker maintains a *UDB*  $(A, B)$  so that after  $\mathfrak{o}$ 's move in round  $r$  there exist integers  $k, \ell$  such that the following properties hold.

- (S1.1)  $D(V \setminus B)$  is an  $\alpha$ -structure of rank  $k$ , such that  $D(V \setminus B)^+ \subseteq A$ ,
- (S1.2)  $D(V \setminus A)$  is an  $\alpha$ -structure of rank  $\ell$ , such that  $D(V \setminus A)^- \subseteq B$ ,
- (S1.3)  $k + \ell = r$ , and
- (S1.4)  $|A| - k = |B| - \ell = r$ .

OBreaker proceeds to Stage II once  $|A| - k = |B| - \ell \geq n/6 \geq n - b$ , that is after round  $\lceil n/6 \rceil$ .

Before the first move in Stage I, note that the properties (S1.1)–(S1.4) hold with  $r = k = \ell = 0$  and  $A = B = \emptyset$ . Now let  $e = (v, w)$  be the arc OMaker directs in a particular round of Stage I. Assume first that  $e \in \mathcal{A}(D(V \setminus B))$ . Then OBreaker adds  $e$  to the  $\alpha$ -structure  $D(V \setminus B)$  by directing the edges  $\{f_1, \dots, f_t\} \subseteq \mathcal{A}(D(V \setminus B))$  given by Lemma 2.8. If  $v \in V \setminus (A \cup B)$  let  $x \in V \setminus (A \cup B \cup \{v\})$ . Then OBreaker directs all edges  $(v, y)$  and  $(x, y)$  for  $y \in B$  and

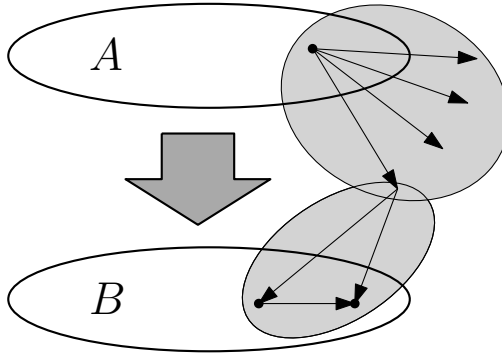


Figure 3: The structure of the digraph OBreaker maintains.

sets  $A := A \cup \{v, x\}$ . (Note that we might delete a vertex from the  $\alpha$ -structure on  $V \setminus A$ , but by Lemma 2.3 this does not affect (S1.2).) Otherwise  $v \in A$  already, so OBreaker picks two arbitrary new vertices  $v', x \in V \setminus (A \cup B)$ , directs all edges  $(v', y)$  and  $(x, y)$  for  $y \in B$  and sets  $A := A \cup \{v', x\}$ . In both cases,  $\tilde{o}$  picks an arbitrary element  $y' \in V \setminus (A \cup B)$ , directs all edges  $(x', y')$  for  $x' \in A$ , and sets  $B := B \cup \{y'\}$ .

Assume now that  $e \notin \mathcal{A}(D(V \setminus B))$ . Then since  $(A, B)$  is a  $UDB$ ,  $e \in \mathcal{A}(D(V \setminus A))$ . Consider the digraph  $\overleftarrow{D}$ . Note that  $(A', B') := (B, A)$  is a  $UDB$  in  $\overleftarrow{D}$  such that properties (S1.1)-(S1.4) hold (with  $k' := \ell$  and  $\ell' := k$ ), by Lemma 2.2. Moreover,  $\overleftarrow{e} \in \mathcal{A}(\overleftarrow{D}[V \setminus B'])$ . Applying the strategy above to  $\overleftarrow{e}$  and  $\overleftarrow{D}$ , one obtains arcs  $f_1, \dots, f_t$  that OBreaker would direct in the "dual game", plus updates of  $A'$  and  $B'$ . OBreaker now directs the reversed arcs  $\overleftarrow{f}_1, \dots, \overleftarrow{f}_t$  and sets  $A := B'$  and  $B = A'$ .

In **Stage II**, OBreaker stops increasing the values  $|A| - k$  and  $|B| - \ell$ .  $\tilde{O}$  now maintains a  $UDB$   $(A, B)$  and integers  $k, \ell$  such that after each move of OBreaker

$$(S2.1) \quad D(V \setminus B) \text{ is an } \alpha\text{-structure of rank } k, \text{ such that } D(V \setminus B)^+ \subseteq A$$

$$(S2.2) \quad D(V \setminus A) \text{ is an } \alpha\text{-structure of rank } \ell, \text{ such that } D(V \setminus A)^- \subseteq B$$

$$(S2.3) \quad |A| - k, |B| - \ell \geq n/6 \geq n - b.$$

OBreaker proceeds to Stage III as soon as  $A \cup B = V$ .

Again, let  $e = (v, w)$  be the arc OMaker directs in a particular round of Stage II and assume first that  $e \in \mathcal{A}(D(V \setminus B))$ . Then OBreaker adds  $e$  to the  $\alpha$ -structure  $D(V \setminus B)$  using Lemma 2.8. If  $v \in V \setminus (A \cup B)$  then  $\tilde{o}$  directs all edges  $(v, y)$  for  $y \in B$  and sets  $A := A \cup \{v\}$ . Otherwise  $v \in A$  already, so OBreaker picks an arbitrary new vertex  $v' \in V \setminus (A \cup B)$ , directs all edges  $(v', y)$  for  $y \in B$  and sets  $A := A \cup \{v'\}$ .

Assume now that  $e \notin \mathcal{A}(D(V \setminus B))$ . Then, since  $(A, B)$  is a  $UDB$ ,  $e \in \mathcal{A}(D(V \setminus A))$ . Similar to Stage I, OBreaker now uses the strategy for the dual structure and claims the reversed arcs.

In **Stage III**, OBreaker maintains a  $UDB$   $(A, B)$  with  $A \cup B = V$  such that

$$(S3.1) \quad D(A) \text{ forms an } \alpha\text{-structure,}$$

$$(S3.2) \quad D(B) \text{ forms an } \alpha\text{-structure, and}$$

$$(S3.3) \quad |A|, |B| \leq b.$$

Let again  $e = (v, w)$  be the arc OMaker directs in her previous move. Either  $e \in \mathcal{A}(D(A))$  or  $e \in \mathcal{A}(D(B))$ . In the first case, OBreaker adds  $e$  to the  $\alpha$ -structure  $D(A)$  using Lemma 2.8; in the second case, OBreaker adds  $e$  to the  $\alpha$ -structure  $D(B)$ , again by using Lemma 2.8. In case  $t = 0$  in Lemma 2.8, OBreaker does not need to direct any arc to reestablish the properties. Then OBreaker directs an arbitrary edge, say  $e' \in \mathcal{A}(D(A))$  (or in  $\mathcal{A}(D(B))$ ), and adds  $e'$  to the  $\alpha$ -structure using Lemma 2.8.

Let us first remark that if OBreaker can follow the proposed strategy and reestablish the properties of the certain stage in each move, then OMaker can never close a cycle. Indeed, throughout the whole game, OBreaker maintains a  $UDB$   $(A, B)$  such that  $D(V \setminus A)$  and  $D(V \setminus B)$  form  $\alpha$ -structures (cf. (S\*.1) and (S\*.2) of each stage). Moreover, also by (S\*.1)

and  $(S * .2)$  of each stage, at any point during the game we have for any  $(v, w) \in D$  that  $v \in A$  or  $w \in B$ . Suppose at some point, OMaker could close a cycle  $C$  by directing an edge  $e = (v, w)$ . Since  $(A, B)$  is a  $UDB$  and by the previous comment, all edges of  $C$  must lie either completely in  $V \setminus A$  or completely in  $V \setminus B$ . However,  $D(V \setminus A)$  (and  $D(V \setminus B)$ ) is an  $\alpha$ -structure, so by Corollary 2.7, OMaker cannot close a cycle in  $V \setminus A$  (and  $V \setminus B$  respectively).

It remains to prove that OBreaker can follow the proposed strategy without forfeiting the game, that in each round  $\tilde{o}$  has to direct at most  $b$  edges, and that the properties of each stage are reestablished.

Recall that the properties  $(S1.1) - (S1.4)$  hold before the first move in Stage I (for technical reasons we say "after round 0") with  $r = k = \ell = 0$  and  $A = B = \emptyset$ .

Suppose now that for some  $r \geq 0$ , after round  $r$  in **Stage I** the properties  $(S1.1) - (S1.4)$  hold. If  $|A| - k, |B| - \ell \geq n/6$ , then OBreaker proceeds to Stage II, so we can assume  $|A| - k = |B| - \ell < n/6$ . As said previously, since  $(A, B)$  is a  $UDB$ , all the arcs  $(x, y)$  with  $x \in A$  and  $y \in B$  are present in  $D$  already, so OMaker's arc is completely in  $V \setminus A$  or completely in  $V \setminus B$ . Assume first that for OMaker's arc  $e = (v, w)$  it holds that  $e \in \mathcal{A}(D(V \setminus B))$ . Since  $D(V \setminus B)$  forms an  $\alpha$ -structure by  $(S1.1)$ , and by Lemma 2.8, OBreaker can add  $e$  to that  $\alpha$ -structure. By  $(S1.2)$  we have  $D(V \setminus A)^- \subseteq B$  before the update of the sets  $A$  and  $B$ . So for all  $z \in V \setminus (A \cup B)$ , all  $y \in V \setminus A$ , either  $(z, y) \in D$  or  $(z, y) \in \mathcal{A}(D)$ . Similarly, by  $(S1.1)$  we have  $D(V \setminus B)^+ \subseteq A$ . So for all  $z \in V \setminus (A \cup B)$  and all  $x \in V \setminus B$  either  $(x, z) \in D$  or  $(x, z) \in \mathcal{A}(D)$ . So OBreaker can claim all edges  $(v, y)$  (or  $(v', y)$ ) and  $(x, y)$  for  $y \in B$ , and all edges  $(x', y')$  for  $x' \in A$  as requested by the strategy.

By Lemma 2.8, adding  $e$  to the  $\alpha$ -structure in  $V \setminus B$  takes OBreaker at most  $k$  edges to direct. Furthermore, since  $|A| - k, |B| - \ell$ , and  $k + \ell$  are bounded by  $n/6$  (by assumption and  $(S1.4)$ ), the strategy asks OBreaker to direct at most

$$k + 2|B| + |A| + 2 = 2(|B| - \ell) + 2(k + \ell) + (|A| - k) + 2 \leq 5 \left\lfloor \frac{n}{6} \right\rfloor + 2 \leq b$$

edges in one round of Stage I. We need to show that the properties are restored. For ease of notation, let us assume that  $v$  was in  $V \setminus (A \cup B)$ , so OBreaker added  $v$  (and not  $v'$ ) to  $A$ . Let  $f_1, \dots, f_t$  be the arcs OBreaker directs in round  $r + 1$ . Let  $D' = D \cup \{e, f_1, \dots, f_t\}$  be the new digraph after OBreaker's move. It is obvious from the strategy description that after the update the pair  $(A, B)$  forms a  $UDB$  again. In round  $r + 1$ , OBreaker adds  $v$  and some vertex  $x \in V \setminus (A \cup B)$  to  $A$ , and some vertex  $y' \in V \setminus (A \cup B)$  to  $B$ . Therefore, by Lemma 2.3, and by Lemma 2.8,  $D'(V \setminus B)$  is then an  $\alpha$ -structure of rank  $k + 1$ , and  $D'(V \setminus B)^+ \subseteq A$ . So  $(S1.1)$  holds again. Since OBreaker's arcs belong to  $A \times V$ , OBreaker at most deletes vertices from the  $\alpha$ -structure  $D(V \setminus A)$ , so  $(S1.2)$  holds by Lemma 2.3. For  $(S1.3)$  note that after OBreaker's move we have one  $\alpha$ -structure of rank  $k + 1$  and one of rank  $\ell$ . Finally, for  $(S1.4)$  note that the rank of the first  $\alpha$ -structure (with rank  $k$ ) increases by one, while OBreaker adds two vertices to  $A$ . For the second  $\alpha$ -structure we do not change the rank, while we increase  $|B|$  by one. Assume now that  $e \notin \mathcal{A}(D(V \setminus B))$ , that is  $e \in \mathcal{A}(D(V \setminus A))$ . Applying the previous argument to the dual  $\overleftarrow{D}$ , it is clear that OBreaker can follow that strategy and that this strategy restablishes the properties  $(S1.1)-(S1.4)$  for  $D$ , using the self-duality of those properties and of  $\alpha$ -structures (c.f. Proposition 2.2).

For **Stage II**,  $(S1.1)$  and  $(S1.2)$  trivially imply  $(S2.1)$  and  $(S2.2)$ .  $(S2.3)$  follows by assump-

tion of entering Stage II. So assume, the three properties hold before OMaker's move in this stage. Assume first that for OMaker's arc  $e = (v, w)$  it holds that  $e \in \mathcal{A}(D(V \setminus B))$ . As in Stage I, OBreaker can add  $e$  to the  $\alpha$ -structure in  $V \setminus B$ . Similarly as in Stage I, by (S2.2), for all vertices  $z \in V \setminus (A \cup B)$  all  $y \in B$ , either  $(z, y) \in D$  or  $(z, y) \in \mathcal{A}(D)$ . So OBreaker can direct all edges  $(v, y)$ , or  $(v', y)$  respectively for  $y \in B$  as requested by the strategy. Furthermore, the strategy asks him to direct at most

$$|B| + k = |V| - (|A| - k) \leq b$$

edges, by Property (S2.3). Finally, the properties are restored. By Lemma 2.3 and Lemma 2.8,  $D(V \setminus B)$  is again an  $\alpha$ -structure. Since  $v$  is added to  $A$  by directing all edges  $(v, y)$  for  $y \in B$ , (S2.1) follows. Again, (S2.2) follows from Lemma 2.3. For (S2.3), note that only the rank of the  $\alpha$ -structure  $D(V \setminus B)$  increased by one, and OBreaker added exactly one new vertex to  $A$  ( $v$  or  $v'$ ). The case  $e \in \mathcal{A}(D(V \setminus A))$  is analogous due to the duality of the properties.

Finally, it is straight-forward that OBreaker can follow the strategy proposed in **Stage III**. Since OBreaker plays in Stage II until  $A \cup B = V$  (and the sets  $A, B$  indeed enlarge in each round), and by (S2.3) it follows that  $|A|, |B| \leq b$ . He then plays either inside  $A$  or  $B$  according to the strategy given by Lemma 2.8 until all edges are directed. Therefore, in one round, OBreaker needs to direct at most  $|A| - 2 < b$  or  $|B| - 2 < b$  edges to add  $e$  (or  $e'$  respectively) to the  $\alpha$ -structure.

This finishes the proof of Theorem 1.2. □

### 3 The Oriented-cycle game – strict rules

In this section, we give a strategy for OBreaker for the strict Oriented-cycle game when playing with bias  $b$ . We later prove that this strategy constitutes a winning strategy if  $19n/20 \leq b \leq n - 3$ . For  $b \geq n - 2$ , a winning strategy is already given by the trivial strategy mentioned in the introduction. The particular details of our strategy are hidden inside some lemmas (cf. Lemma 3.3, 3.4, 3.7, 3.8, 3.9), which we prove in the subsequent sections.

In our strategy for the monotone rules, OBreaker aims for a  $UDB$   $(A, B)$  as global structure, and handles OMaker's edges locally using  $\alpha$ -structures. The reason for  $\alpha$ -structures being powerful is that OBreaker stops directing further edges in this game once the digraph has the desired local structure. In the strict rules, this is of course no longer possible. Therefore, we introduce a different local structure, which is more robust to adding edges.

We divide the strategy into two stages. Similarly as in the monotone game, in Stage I, OBreaker creates two large buffer sets  $A'$  and  $B'$ , both being independent in  $D$ , such that  $(A', B')$  forms a  $UDB$  in  $D$ . In Stage II, OBreaker maintains two such sets of a certain minimum size.

**Stage I** lasts exactly  $\lfloor n/25 \rfloor$  rounds.

We call the structure of  $D$  that OBreaker maintains during Stage I *riskless* and define it as follows. Following general notation, a *down set* of  $[n]$  is a set  $[k]$ , for some  $k \leq n$ , and an *upset* of  $[n]$  is a set  $[n] \setminus [k]$ , for some  $k \leq n$ .

**Definition 3.1.** A digraph  $D$  is called *riskless* of rank  $r$  if there is a UDB  $(A, B)$  with partitions  $A = A_S \cup A_0$  and  $B = B_S \cup B_0$  such that the following properties hold:

- (R1) Size:  $||A| - |B|| \leq 1$  and  $|A_S| = |B_S| = r$ .
- (R2) Structure of  $A_S$  and  $B_S$ : There exist enumerations  $A_S = \{v_1, \dots, v_r\}$  and  $B_S = \{w_1, \dots, w_r\}$  such that
  - (R2.1)  $(v_1, \dots, v_r)$  and  $(w_1, \dots, w_r)$  induce transitive tournaments in  $D$ .
  - (R2.2) For all  $z \in A_0 \cup V \setminus (A \cup B)$ :  $\{i : (v_i, z) \in D\}$  is a down set of  $[r]$ .
  - (R2.3) For all  $z \in B_0 \cup V \setminus (A \cup B)$ :  $\{i : (z, w_i) \in D\}$  is an upset of  $[r]$ .
- (R3) Stars: For every  $1 \leq i \leq r$ :
  - (R3.1)  $e_D(v_i, A_0) \leq r + 1 - i$  and  $e_D(v_i, V \setminus (A \cup B)) \leq \max(|A|, |B|)$ .
  - (R3.2)  $e_D(B_0, w_i) \leq i$  and  $e_D(V \setminus (A \cup B), w_i) \leq \max(|A|, |B|)$ .
- (R4) Edge set:  $D = D(A, B) \cup D(A_S, V \setminus B) \cup D(V \setminus A, B_S)$ .

An illustration of the properties can be found in Figure 4. Observe that the definition of riskless is self-dual in the following sense. Recall that  $\overleftarrow{D}$  denotes the digraph obtained by reversing all arcs of a digraph  $D$ .

**Observation 3.2.** A digraph  $D$  is a riskless digraph of rank  $r$  with UDB  $(A, B)$  if and only if  $\overleftarrow{D}$  is a riskless digraph of rank  $r$  with UDB  $(B, A)$ .

Note that the empty graph, which is present before the first round of the game (for technical reasons we say "after round 0"), is riskless of rank 0. For some  $0 \leq r \leq n/25 - 1$ , after round  $r$ , assume  $D$  is riskless of rank  $r$  with UDB  $(A, B)$ . Note that, since OBreaker directs exactly  $b$  edges in each round and OMaker directs exactly one edge in each round,  $|D| = r(b + 1)$ . Let  $e = (v, w)$  be the arc OMaker directs in round  $r + 1$ . Suppose first that  $e \in \mathcal{A}(D(V \setminus B))$ . OBreaker now follows a *base strategy* and chooses *at most*  $b$  arcs given by the following lemma to restore the properties of a riskless digraph, increasing the rank by one.

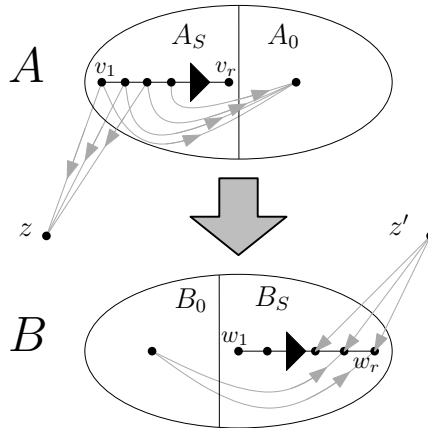


Figure 4: The structure of a *riskless* digraph.

**Lemma 3.3** (Base strategy I). *Let  $19n/20 \leq b \leq n-3$ . For a non-negative integer  $r \leq n/25-1$ , let  $D$  be a digraph which is riskless of rank  $r$  with  $UDB (A, B)$  as given in Definition 3.1. Assume that  $|D| = r(b+1)$ . Let  $e \in \mathcal{A}(D(V \setminus B))$  be an available arc in  $V \setminus B$ . Then there exist at most  $b$  available arcs  $f_1, \dots, f_t \in \mathcal{A}(D+e)$  such that  $D' := D \cup \{e, f_1, \dots, f_t\}$  is a riskless digraph of rank  $r+1$ . Moreover,  $e_{D'}(V \setminus (A' \cup B'), w'_1) = 0$ , where  $(A', B')$  is the underlying  $UDB$  of  $D'$ , with  $B' = B'_S \cup B'_0$  and  $B'_S = \{w'_1, \dots, w'_{r+1}\}$  as in Definition 3.1.*

The property  $e_{D'}(V \setminus (A' \cup B'), w'_1) = 0$  is useful to accommodate OBreaker's remaining arcs that still needs to direct in round  $r+1$ . Let  $t$  be the number of edges OBreaker directs to follow the base strategy in round  $r+1$ . He then adds  $b-t$  further arcs using the following lemma.

**Lemma 3.4** (Add-edges strategy I). *Let  $19n/20 \leq b \leq n-3$ . For a non-negative integer  $r \leq n/25-1$ , let  $D$  be a digraph which is riskless of rank  $r+1$  with  $UDB (A, B)$  as given in Definition 3.1. Assume that  $r(b+1) \leq |D| \leq (r+1)(b+1) < \binom{n}{2}$ . Let  $w_1$  be the top vertex in the tournament inside  $B$ , as given in Property (R2), and assume that  $e_D(V \setminus (A \cup B), w_1) = 0$  holds. Then OBreaker can direct a set of  $(r+1)(b+1) - |D| \leq b$  available arcs  $\mathcal{F} \subseteq \mathcal{A}(D)$  such that  $D' := D \cup \mathcal{F}$  is a riskless digraph of rank  $r+1$ .*

Assume now that  $e = (v, w) \notin \mathcal{A}(D(V \setminus B))$ . Then we know that  $e \in \mathcal{A}(D(V \setminus A))$ , since  $(A, B)$  forms a  $UDB$ . By Observation 3.2,  $\overleftarrow{D}$  is also riskless of rank  $r$  with  $UDB (B, A)$ . So, applying Lemma 3.3 and Lemma 3.4 we thus can find a set  $\mathcal{F}$  of exactly  $b$  available arcs such that  $\overleftarrow{D} \cup \{\overleftarrow{e}\} \cup \mathcal{F}$  is riskless of rank  $r+1$ . But then again the dual  $D \cup \{e\} \cup \overleftarrow{\mathcal{F}}$  is riskless of rank  $r+1$ . OBreaker thus chooses the  $b$  arcs  $\overleftarrow{\mathcal{F}}$  in this case.

**Stage II** starts in round  $\lfloor n/25 \rfloor + 1$ .

The structure that OBreaker now aims to maintain, is similar to the one given for Stage I. The most important difference is that from now on we partition the sets of the  $UDB$  further to distinguish the vertices according to their chance to become part of a directed cycle. We have partitions  $A = A_D \cup A_{AD} \cup A_S \cup A_0$  and  $B = B_D \cup B_{AD} \cup B_S \cup B_0$ . (The subscripts stand for *Dead*, *Almost Dead* and *Star*.)  $A_0$  and  $B_0$  form the aforementioned buffer sets together with all dead vertices.

**Definition 3.5.** *A digraph  $D$  on  $n$  vertices is called protected if there is a  $UDB (A, B)$  with partitions  $A = A_D \cup A_{AD} \cup A_S \cup A_0$  and  $B = B_D \cup B_{AD} \cup B_S \cup B_0$  such that the following properties hold:*

- (P1) Sizes:  $|A|, |B| \geq n/10 + 1$ , and  $|A_D \cup A_0|, |B_D \cup B_0| \geq n/10$ .
- (P2) Dead vertices: The pairs  $(A_D, V \setminus A_D)$  and  $(V \setminus B_D, B_D)$  form  $UDB'$ s,  $D(A_D)$  and  $D(B_D)$  are transitive tournaments.
- (P3) Almost-dead vertices: The pairs  $(A_{AD}, V \setminus A)$  and  $(V \setminus B, B_{AD})$  form  $UDB'$ s.
- (P4) Structure of  $A_{AD} \cup A_S$  and  $B_{AD} \cup B_S$ : There exist integers  $k_1, \ell_1 \geq 0$  and  $0 \leq k_2, \ell_2 \leq n/25$  and enumerations

$$A_{AD} = \{v_1, \dots, v_{k_1}\}, A_S = \{v_{k_1+1}, \dots, v_{k_1+k_2}\} \text{ and} \\ B_S = \{w_1, \dots, w_{\ell_2}\}, B_{AD} = \{w_{\ell_2+1}, \dots, w_{\ell_2+\ell_1}\} \text{ such that}$$

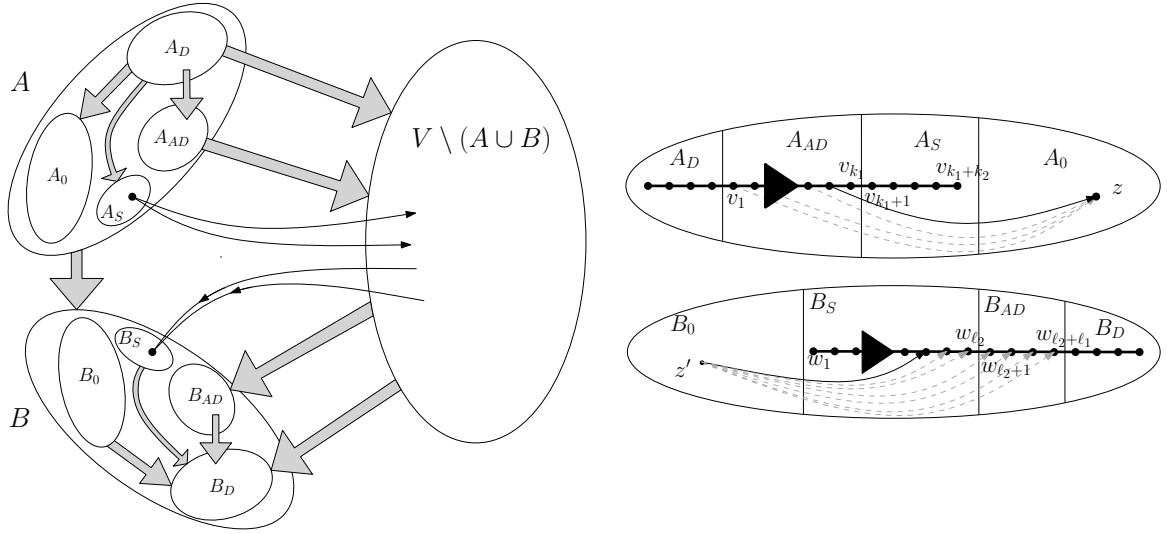


Figure 5: The global and local structure of a protected digraph.

(P4.1)  $(v_1, \dots, v_{k_1+k_2})$  and  $(w_1, \dots, w_{\ell_1+\ell_2})$  induce transitive tournaments.

(P4.2) For all  $z \in A_0 \cup V \setminus (A \cup B)$ :  $\{i : (v_i, z) \in D\}$  is a down set of  $[k_1 + k_2]$

(P4.3) For all  $z \in B_0 \cup V \setminus (A \cup B)$ :  $\{i : (z, w_i) \in D\}$  is an upset of  $[\ell_1 + \ell_2]$ .

(P5) Stars:

(P5.1) For all  $1 \leq i \leq k_2$ :  $e(v_{k_1+i}, A_0) \leq n/25 + 1 - i$ .

(P5.2) For all  $1 \leq i \leq \ell_2$ :  $e(B_0, w_i) \leq n/25 - \ell_2 + i$ .

(P6) Edge set:  $E(D) = D(A, B) \cup D(A \setminus A_0, V \setminus B) \cup D(V \setminus A, B \setminus B_0)$ .

An illustration of a protected digraph can be found in Figure 5.

Again, the definition of protected is self-dual in the following sense.

**Observation 3.6.** A digraph  $D$  is a protected digraph with  $UDB(A, B)$  if and only if  $\overleftarrow{D}$  is a protected digraph with  $UDB(B, A)$ .

First, we claim that after OBReaker's last move in Stage I, the digraph  $D$  is protected. Note that, on the assumption of Lemma 3.3 and Lemma 3.4,  $D$  is riskless of rank  $\lfloor n/25 \rfloor$ .

**Lemma 3.7.** Let  $n$  be large enough, and let  $19n/20 \leq b \leq n - 3$ . Let  $D$  be a digraph on  $n$  vertices which is riskless of rank  $r = \lfloor n/25 \rfloor$ , and assume that  $|D| = r(b + 1)$ . Then  $D$  is protected.

For some  $r \geq \lfloor n/25 \rfloor$ , after round  $r$ , assume  $D$  is protected with  $UDB(A, B)$ . Let  $e = (v, w)$  be the arc OMaker directs in round  $r + 1$ . Again, suppose first that  $e \in \mathcal{A}(D(V \setminus B))$ . OBReaker now follows a *base strategy* and chooses at most  $b$  arcs given by the following lemma to restore the properties of a protected digraph.

**Lemma 3.8** (Base strategy II). *Let  $D$  be a digraph which is protected and let  $e = (v, w) \in \mathcal{A}(D(V \setminus B))$  be an available arc in  $V \setminus B$ . Then there exist at most  $b$  available arcs  $f_1, \dots, f_t \in \mathcal{A}(D + e)$  such that  $D' := D \cup \{e, f_1, \dots, f_t\}$  is protected.*

Let  $t$  be the number of edges OBreaker directs to follow the base strategy in round  $r + 1$ . He then adds  $b - t$  further edges using the following lemma iteratively (unless there are fewer than  $b - t$  available edges whence he directs all remaining edges according to the following lemma).

**Lemma 3.9** (Add-edges strategy II). *Let  $D$  be a digraph which is protected. Then, unless  $D$  is a tournament on  $n$  vertices, there exists an available arc  $f \in \mathcal{A}(D)$  such that  $D + f$  is protected.*

Assume now that  $e = (v, w) \notin \mathcal{A}(D(V \setminus B))$ . Then we know that  $e \in \mathcal{A}(D(V \setminus A))$ , since  $(A, B)$  forms a  $UDB$ . By Observation 3.6,  $\overleftarrow{D}$  is also protected with  $UDB(B, A)$ . So, applying Lemma 3.8 and Lemma 3.9 we thus can find a set  $\mathcal{F}$  of exactly  $b$  (at most  $b$  in the last round of the game) available arcs such that  $\overleftarrow{D} \cup \{\overleftarrow{e}\} \cup \mathcal{F}$  is protected. But then again the dual  $D \cup \{e\} \cup \overleftarrow{\mathcal{F}}$  is protected. OBreaker thus chooses the  $b$  arcs  $\overleftarrow{\mathcal{F}}$  in this case.

The strategy for OBreaker is given implicitly in Lemma 3.3, 3.4, 3.8 and 3.9. These lemmas, together with Lemma 3.7, also contain the proof that OBreaker can follow that strategy. We prove Lemma 3.3 and Lemma 3.4 in the next section; Lemma 3.7, Lemma 3.8 and Lemma 3.9 are proved in Section 5.

We finish this section with the proof that the proposed strategy constitutes a winning strategy. In Stage I, Lemma 3.3 and Lemma 3.4 guarantee that the digraph  $D$  is riskless of rank  $r$  after OBreaker's move in a particular round  $r$ . Furthermore, after round  $r = \lfloor n/25 \rfloor$ ,  $D$  is protected by Lemma 3.7. Then, in Stage II, Lemma 3.8 and Lemma 3.9 guarantee that the digraph  $D$  is protected after each move of OBreaker. Therefore, to show that OMaker can never close a cycle, it is enough to prove the following.

**Lemma 3.10.** *If  $D$  is a protected digraph, then  $D$  is acyclic.*

*Proof.* Let  $D$  be a protected digraph with  $UDB(A, B)$ , and suppose there is a directed cycle  $C$  in  $D$ . By property (P6), for each  $(v, w) \in D$ , we have  $v \in A$  or  $w \in B$ . Therefore, the underlying edges of  $C$  either only contain vertices from  $A$  or only contain vertices from  $B$ . By Observation 3.6, we may assume without loss of generality that  $C \subseteq D(A)$ . Again by Property (P6),  $C$  must use only vertices from  $A \setminus A_0$ . However, by Property (P2) and Property (P4.1),  $A \setminus A_0$  induces a transitive tournament, and thus does not contain a directed cycle, a contradiction.  $\square$

## 4 Strict rules - Stage I

In the following we prove Lemma 3.3 and Lemma 3.4. First, let us pin down a proposition which we shall use frequently.

**Proposition 4.1.** *Let  $n$  be large enough, and let  $b \leq n - 3$  and  $r \leq n/25$ . Let  $D$  be a riskless digraph of rank  $r$ , with underlying  $UDB(A, B)$ , such that  $|D| \leq r(b + 1)$ . Then*



(i)  $|A|, |B| < n/5 + 1$  and  $|V \setminus (A \cup B)| > 3n/5 - 1$ .

(ii)  $|X_A|, |Y_B| > 2n/5 - 2$  where  $X_A = \{z \in V \setminus (A \cup B) : e_D(A, z) = 0\}$  and  $Y_B = \{z \in V \setminus (A \cup B) : e_D(z, B) = 0\}$ .

*Proof.* Note first that since  $(A, B)$  is a *UDB*  $|A| \cdot |B| \leq |D| \leq r(b+1) < n^2/25$ , by assumption on  $r$  and  $b$ . Since  $||A| - |B|| \leq 1$  (by Property (R1)), it follows that  $\max(|A|, |B|) < n/5 + 1$ , and that  $|A| + |B| < 2n/5 + 1$ . Therefore,  $|V \setminus (A \cup B)| > 3n/5 - 1$ . Let  $\overline{X_A} := \{z \in V \setminus (A \cup B) : e_D(A, z) > 0\}$ . Then, by Property (R4) and (R2.2),

$$\begin{aligned} \overline{X_A} &= \{z \in V \setminus (A \cup B) : (x, z) \in D \text{ for some } x \in A\} \\ &= \{z \in V \setminus (A \cup B) : (x, z) \in D \text{ for some } x \in A_S\} \\ &= \{z \in V \setminus (A \cup B) : (v_1, z) \in D\}. \end{aligned}$$

So, by Property (R3.1) and (i),  $|\overline{X_A}| \leq \max(|A|, |B|) < n/5 + 1$ , and hence,

$$|X_A| = |V \setminus (A \cup B)| - |\overline{X_A}| > \frac{2n}{5} - 2.$$

Similarly,  $|Y_B| > 2n/5 - 2$ . □

Now, we prove Lemma 3.3. It ensures that OBreaker has a strategy to reestablish the properties of a riskless graph throughout Stage I.

*Proof of Lemma 3.3.* For  $e \in \mathcal{A}(D(V \setminus B))$  given by the lemma, let  $v = e^+$  and  $w = e^-$ . At first, let us fix distinct vertices  $x_A, y_B \in V \setminus (A \cup B \cup \{v, w\})$  with  $e_D(A, x_A) = 0$  and  $e_D(y_B, B) = 0$ . Note that Proposition 4.1 guarantees their existence since  $r \leq n/25$  and  $b \leq n - 3$ . Define  $u_S \in A_0 \cup V \setminus (A \cup B)$  and  $u_A \in V \setminus (A \cup B)$  by

$$u_S := \begin{cases} x_A & \text{if } v \in A_S \\ v & \text{if } v \notin A_S \end{cases} \quad \text{and} \quad u_A := \begin{cases} x_A & \text{if } v \in A \\ v & \text{if } v \notin A. \end{cases}$$

Our goal is to add  $u_S$  to the set  $A_S$  of star centers,  $u_A$  to  $A$  and  $y_B$  to  $B$ . Note that the two vertices  $u_s$  and  $u_A$  are equal unless  $v \in A_0$ . Moreover, let

$$\ell := \begin{cases} \min(i : (v_i, v) \notin D) & \text{if minimum exists} \\ r + 1 & \text{otherwise} \end{cases}$$

and observe that  $v_\ell = v$  if  $v \in A_S$  (by Property (R2.1)). Set

$$v'_i := \begin{cases} v_i & 1 \leq i \leq \ell - 1 \\ u_S & i = \ell \\ v_{i-1} & \ell + 1 \leq i \leq r + 1 \end{cases} \quad \text{and} \quad w'_i := \begin{cases} y_B & i = 1 \\ w_{i-1} & 2 \leq i \leq r + 1. \end{cases}$$

These vertices are used to form the new centers of the stars in  $A_S$  and  $B_S$ . Now, choose  $\{f_1, \dots, f_t\}$  to be  $\{f_1, \dots, f_t\} = (\mathcal{F}_1 \cup \dots \cup \mathcal{F}_7) \cap \mathcal{A}(D)$ , where

$$\begin{aligned}\mathcal{F}_1 &:= \{(u_A, y) : y \in B\} \\ \mathcal{F}_2 &:= \{(x, w'_1) : x \in A \cup \{u_A\}\} \\ \mathcal{F}_3 &:= \{(w'_1, w'_i) : 2 \leq i \leq r+1\} \\ \mathcal{F}_4 &:= \{(v'_i, w) : 1 \leq i \leq \ell\} \\ \mathcal{F}_5 &:= \{(v'_i, v'_\ell) : 1 \leq i \leq \ell-1\} \\ \mathcal{F}_6 &:= \{(v'_\ell, v'_i) : \ell+1 \leq i \leq r+1\} \\ \mathcal{F}_7 &:= \{(v'_\ell, z) : z \in V \setminus (A \cup B) \cup A_0 \text{ and } (v'_{\ell+1}, z) \in D\},\end{aligned}$$

where we use the convention that  $\mathcal{F}_7 = \emptyset$  if  $\ell = r+1$  (and thus  $v'_{\ell+1}$  does not exist). The illustration of these arc sets can be found in Figure 6.

To show that this choice of arcs is suitable for the lemma, we first show that  $f_i \neq \overleftarrow{e}$  for all  $1 \leq i \leq t$ , that  $\mathcal{F}_1 \cup \dots \cup \mathcal{F}_7 \subseteq D \cup \mathcal{A}(D)$ , and that  $t \leq b$ . Note that  $\mathcal{F}_i \subseteq D \cup \mathcal{A}(D)$  implies that  $\mathcal{F}_i \subseteq D'$ , where  $D' = D \cup \{e, f_1, \dots, f_t\}$ . We use this information to show that  $D'$  is riskless of rank  $r+1$  with some UDB  $(A', B')$ . Finally, we show that  $e_{D'}(V \setminus (A' \cup B'), w'_1) = 0$ , where  $w'_1$  is the top vertex in the tournament in  $B'$ , given by property (R2.1).

For the first part,  $\overleftarrow{e} = (w, v) \notin \mathcal{F}_1$  since by assumption  $v \notin B$ ,  $\overleftarrow{e} \notin \mathcal{F}_2 \cup \mathcal{F}_3$  by choice of  $w'_1 = y_B$ , and  $\overleftarrow{e} \notin \mathcal{F}_4$  since  $v \neq w$ . Assume now that  $(w, v) = (v'_i, v'_\ell) \in \mathcal{F}_5$  for some  $1 \leq i \leq \ell-1$ . Then,  $(w, v) = (v_i, v) \in D$  by definition of  $\ell$ , a contradiction to  $e = (v, w) \in \mathcal{A}(D(V \setminus B))$ . So,  $\overleftarrow{e} \notin \mathcal{F}_5$ . For  $\mathcal{F}_6$ , assume that  $(w, v) = (v'_\ell, v'_i) \in \mathcal{F}_6$  for some  $\ell+1 \leq i \leq r+1$ . Then  $v \neq v'_\ell$ , so  $v'_\ell = u_S = x_A \neq w$  by definition of  $x_A$ , a contradiction. So,  $\overleftarrow{e} \notin \mathcal{F}_6$ . Finally, if  $(w, v) \in \mathcal{F}_7$  then  $v \in V \setminus (A \cup B) \cup A_0$ , so  $w = v'_\ell = u_S = v$  by definition of  $u_S$ , a contradiction since  $(v, w) \in \mathcal{A}(D)$  and thus, it is not a loop. Hence,  $\overleftarrow{e} \notin \mathcal{F}_7$ , as well.

To see that  $\mathcal{F}_1 \subseteq D \cup \mathcal{A}(D)$  note that  $(y, u_A) \notin D$  for all  $y \in B$ , since  $u_A \in V \setminus (A \cup B)$  and by Property (R4). Similarly,  $\mathcal{F}_2 \cup \mathcal{F}_3 \subseteq D \cup \mathcal{A}(D)$  since  $w'_1 = y_B \in V \setminus (A \cup B)$  and by Property (R4). Assume now that  $(w, v'_i) \in D$  for some  $1 \leq i \leq \ell$ . Then  $w = v_j \in A_S$  for some  $1 \leq j \leq r$ , by Property (R4). Moreover,  $v'_i \neq v$ , since  $e = (v, w) \in \mathcal{A}(D(V \setminus B))$ , and  $v'_i \neq x_A$  by choice of  $x_A$  and Property (R4). This yields  $v'_i \neq u_S$  and therefore  $i \neq \ell$ .

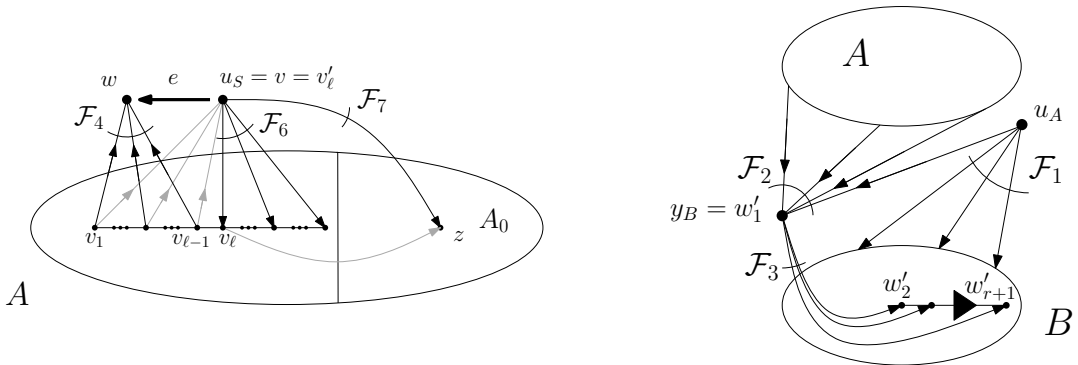


Figure 6: In case  $v \in V \setminus (A \cup B)$ , the family  $\mathcal{F}_6$  includes  $v$  into the tournament in  $A$  ( $\mathcal{F}_5 \subseteq D$  already and thus grey),  $\mathcal{F}_4$  and  $\mathcal{F}_7$  restore (R2.2). The family  $\mathcal{F}_1$  adds  $u_A$  to  $A$ ,  $\mathcal{F}_2$  adds  $y_B$  to  $B$ , and  $\mathcal{F}_3$  adds  $y_B$  to the tournament in  $B$ .

But then  $v'_i = v_i$ , and  $j < i \leq \ell - 1$  by Property (R2.1). By definition of  $\ell$  we conclude  $(w, v) = (v_j, v) \in D$ , in contradiction to  $e \in \mathcal{A}(D(V \setminus B))$ . Thus,  $\mathcal{F}_4 \subseteq D \cup \mathcal{A}(D)$ . For  $\mathcal{F}_5$ , we note that  $(v'_\ell, v'_i) = (u_S, v_i) \notin D$  for all  $1 \leq i < \ell$  by Property (R4) since  $u_S \notin A_S$  and  $v_i \notin B$ . Now, assume that  $(v'_i, v'_\ell) = (v_{i-1}, u_S) \in D$  for some  $\ell + 1 \leq i \leq r + 1$ . Then  $u_S \neq x_A$  by the choice of  $x_A$ . It follows  $(v_{i-1}, v) = (v_{i-1}, u_S) \in D$ . But then, by definition of  $\ell$  and (R2.2), we obtain  $i - 1 < \ell$ , a contradiction. Thus  $\mathcal{F}_6 \subseteq D \cup \mathcal{A}(D)$ . Finally,  $\mathcal{F}_7 \subseteq D \cup \mathcal{A}(D)$  since for all  $z \in V \setminus (A \cup B) \cup A_0$  we have that  $(z, v'_\ell) \notin D$ , by Property (R4) and since  $v'_\ell = u_S \notin B$ .

To bound  $t$  by the bias  $b$  we note that

$$\begin{aligned} t &\leq \sum_{i=1}^7 |\mathcal{F}_i| \leq |B| + (|A| + 1) + r + (2\ell - 1) + (r - \ell + 1) + e_D(v'_{\ell+1}, V \setminus (A \cup B) \cup A_0) \\ &= |A| + |B| + 2r + \ell + 1 + e_D(v_\ell, V \setminus (A \cup B)) + e_D(v_\ell, A_0) \\ &\leq |A| + |B| + 2r + \ell + 1 + \max(|A|, |B|) + (r + 1 - \ell) \\ &\leq 3 \max(|A|, |B|) + 3r + 2 \end{aligned}$$

where the third inequality follows from Property (R3.1).

Now, by Proposition 4.1 (i) and since  $r \leq n/25 - 1$ , it follows that for  $n > 8$ ,

$$t \leq \frac{3n}{5} + \frac{3n}{25} + 2 \leq \frac{19n}{20} \leq b.$$

We now show that  $D' = D \cup \{e, f_1, \dots, f_t\}$  is a riskless digraph of rank  $r + 1$ .

Set  $A'_S := \{v'_1, \dots, v'_{r+1}\} = A_S \cup \{u_S\}$ ,  $A'_0 := (A_0 \cup \{u_A\}) \setminus \{u_S\}$  and  $A' := A \cup \{u_A\}$ , and  $B'_S := \{w'_1, \dots, w'_{r+1}\} = B_S \cup \{y_B\}$ ,  $B'_0 := B_0$  and  $B' = B \cup \{y_B\}$ . We claim that  $(A', B')$  is a  $UDB$  in  $D'$  with partitions  $A' = A'_0 \cup A'_S$  and  $B' = B'_0 \cup B'_S$  such that the properties (R1)–(R4) are satisfied for  $r + 1$  and such that  $e_{D'}(V \setminus (A' \cup B'), w'_1) = 0$ . Checking the properties is straight-forward and we want to guide the reader back to Figure 4 and Figure 6 before plunging into the technical details that follow.

Since  $(A, B)$  is a  $UDB$  in  $D$  and  $D \cup \mathcal{F}_1 \cup \mathcal{F}_2 \subseteq D'$ ,  $(A', B')$  forms a  $UDB$  in  $D'$ . For Property (R1), note that  $|A'| = |A| + 1$ ,  $|B'| = |B| + 1$  and  $|A'_S| = |B'_S| = r + 1$ .

For Property (R2.1), note first that  $A_S = A'_S \setminus \{v'_\ell\}$  induces a transitive tournament in  $D \subseteq D'$ . Furthermore, for all  $i < \ell$  we have that  $(v'_i, v'_\ell) \in \mathcal{F}_5 \subseteq D'$ , and for all  $i > \ell$  we have that  $(v'_\ell, v'_i) \in \mathcal{F}_6 \subseteq D'$ . Hence,  $(v'_1, \dots, v'_{r+1})$  induces a transitive tournament in  $D'$ . Furthermore,  $B_S = B'_S \setminus \{w'_1\}$  induces a transitive tournament in  $D \subseteq D'$ , and for all  $2 \leq i \leq r + 1$ , we have that  $(w'_1, w'_i) \in \mathcal{F}_3 \subseteq D'$ . Hence,  $(w'_1, \dots, w'_{r+1})$  induces a transitive tournament in  $D'$ .

For (R2.2), let  $z \in A'_0 \cup V \setminus (A' \cup B')$ . Then note that only arcs from  $D \cup \{e\} \cup \mathcal{F}_4 \cup \mathcal{F}_7$  contribute to the set  $\{i : (v'_i, z) \in D'\}$ . Moreover,  $D(v'_\ell, A'_0 \cup V \setminus (A' \cup B')) = \emptyset$  by Property (R4) and since  $v'_\ell \in A_0 \cup V \setminus (A \cup B)$ . Note also that  $\{i : (v_i, z) \in D\}$  is a down set of  $[r]$  and the relative order of the  $v'_i$  for  $i \neq \ell$  did not change. If  $z \neq w$ , then the arcs from  $\mathcal{F}_7$  reestablish the down-set property for  $D'$ . If  $z = w$ , then the arcs from  $\mathcal{F}_4 \cup \{e\}$  reestablish the down-set property for  $D'$ .

For (R2.3), let  $z \in B'_0 \cup V \setminus (A' \cup B')$ . Then note that only arcs from  $D$  contribute to the set  $\{i : (z, w'_i) \in D'\}$ . Moreover,  $\{i : (z, w_i) \in D\}$  is an upset of  $[r]$  by assumption. Further, note that  $w'_{i+1} = w_i$  for every  $1 \leq i \leq r$ , and that  $(z, w'_1) = (z, y_B) \notin D$  by Property (R4) and since  $y_B \in V \setminus (A \cup B)$ . Thus,  $\{i : (z, w'_i) \in D'\}$  is an upset of  $[r + 1]$ .

For (R3.1), let first  $1 \leq i \leq \ell - 1$ . Then only the arcs in  $D(v_i, A_0) \cup \mathcal{F}_4$  contribute to  $D'(v'_i, A'_0)$ ; and only the arcs in  $D(v_i, V \setminus (A \cup B)) \cup \mathcal{F}_4$  contribute to  $D'(v'_i, V \setminus (A' \cup B'))$ . Therefore,  $e_{D'}(v'_i, A'_0) \leq e_D(v_i, A_0) + 1 \leq (r+1) + 1 - i$  and  $e_{D'}(v'_i, V \setminus (A' \cup B')) \leq e_D(v_i, V \setminus (A \cup B)) + 1 \leq \max(|A|, |B|) + 1 = \max(|A'|, |B'|)$ .

Now, let  $i = \ell$ . Observe that  $e_D(v'_\ell, A_0 \cup V \setminus (A \cup B)) = 0$  by (R4) and since  $v'_\ell \in A_0 \cup V \setminus (A \cup B)$ . So, only the arcs in  $\mathcal{F}_7 \cup \{e\}$  contribute to  $D'(v'_\ell, A'_0)$  and  $D'(v'_\ell, V \setminus (A' \cup B'))$ . Therefore,  $e_{D'}(v'_\ell, A'_0) \leq e_D(v_\ell, A_0) + 1 \leq (r+1) + 1 - \ell$  and  $e_{D'}(v'_\ell, V \setminus (A' \cup B')) \leq e_D(v_\ell, V \setminus (A \cup B)) + 1 \leq \max(|A|, |B|) + 1 = \max(|A'|, |B'|)$ .

Finally, let  $\ell + 1 \leq i \leq r + 1$ . Then  $D'(v'_i, A'_0) = D(v_{i-1}, A_0)$  and  $D'(v'_i, V \setminus (A' \cup B')) \subseteq D(v_{i-1}, V \setminus (A \cup B))$ . Hence,  $e_{D'}(v'_i, A'_0) \leq (r + 1) + 1 - i$  and  $e_{D'}(v'_i, V \setminus (A' \cup B')) \leq \max(|A'|, |B'|)$ .

For (R3.2), first let  $2 \leq i \leq r + 1$ . Then only the arcs in  $D(B_0, w_{i-1})$  contribute to  $D'(B'_0, w'_i)$ ; and only the arcs in  $D(V \setminus (A \cup B), w'_i)$  contribute to  $D'(V \setminus (A' \cup B'), w'_i)$ . Therefore we have  $e_{D'}(B'_0, w'_i) \leq i - 1$  and  $e_{D'}(V \setminus (A' \cup B'), w'_i) \leq \max(|A|, |B|) \leq \max(|A'|, |B'|)$ .

Now, for  $i = 1$  we have  $w'_1 = y_B \in V \setminus (A \cup B)$ . Similarly, only the arcs in  $D(B_0, y_B)$  contribute to  $D'(B'_0, w'_1)$ ; and only the arcs in  $D(V \setminus (A \cup B), y_B)$  contribute to  $D'(V \setminus (A' \cup B'), w'_1)$ . Then by Property (R4) for the digraph  $D$  we know  $e_{D'}(B'_0, w'_1) = e_D(B_0, y_B) = 0$  and analogously  $e_{D'}(V \setminus (A' \cup B'), w'_1) = 0$ .

For (R4), note that it is enough to prove that  $D' \subseteq D'(A', B') \cup D'(A'_S, V) \cup D'(V, B'_S)$ . This indeed holds, since

$$\begin{aligned} D(A, B) \cup \mathcal{F}_1 \cup \mathcal{F}_2 &\subseteq D'(A', B'), \\ D(A_S, V \setminus B) \cup \mathcal{F}_4 \cup \mathcal{F}_5 \cup \mathcal{F}_6 \cup \mathcal{F}_7 &\subseteq D'(A'_S, V), \text{ and} \\ D(V \setminus A, B_S) \cup \mathcal{F}_3 &\subseteq D'(V, B'_S). \end{aligned} \quad \square$$

Next, we prove Lemma 3.4 which ensures that OBreaker can add arcs to a riskless graph without destroying its structural properties.

*Proof of Lemma 3.4.* We first provide OBreaker with a strategy to direct  $(r+1)(b+1) - |D|$  available arcs, then we show that OBreaker can follow that strategy, and that the resulting digraph  $D'$  is riskless of rank  $r+1$ .

Initially, set  $t = (r+1)(b+1) - |D|$  and let  $\mathcal{F} := \emptyset$  be the (dynamic) set of OBreaker's edges. We proceed iteratively: As long as  $t \geq \max(|A|, |B|)$ , OBreaker enlarges  $A_0$  and  $B_0$  (and thus  $A$  and  $B$ ) alternately. As soon as  $t < \max(|A|, |B|)$ , he fills up the stars with centres  $w_i$ ,  $1 \leq i \leq r$ , starting with  $w_{r+1}$ .

**Step 1:**  $t \geq \max(|A|, |B|)$ . If  $|B| = |A| - 1$ , then let  $y_B \in V \setminus (A \cup B)$  such that  $e_{D \cup \mathcal{F}}(y_B, B) = 0$ . For all  $x \in A$ , if the arc  $(x, y_B) \notin D \cup \mathcal{F}$ , OBreaker directs  $(x, y_B)$ , decreases  $t$  by one and updates  $\mathcal{F} := \mathcal{F} \cup \{(x, y_B)\}$ . Set  $B := B \cup \{y_B\}$ ,  $B_0 := B_0 \cup \{y_B\}$  and repeat Step 1.

If  $|B| \geq |A|$ , then let  $x_A \in V \setminus (A \cup B)$  such that  $e_{D \cup \mathcal{F}}(A, x_A) = 0$ . For all  $y \in B$ , if the arc  $(x_A, y) \notin D \cup \mathcal{F}$ , OBreaker directs  $(x_A, y)$ , updates  $\mathcal{F} := \mathcal{F} \cup \{(x_A, y)\}$  and decreases  $t$  by one. Set  $A := A \cup \{x_A\}$ ,  $A_0 := A_0 \cup \{x_A\}$  and repeat Step 1.

**Step 2:**  $t < \max(|A|, |B|)$ . If  $t = 0$ , there is nothing to do. Otherwise, OBreaker proceeds as follows.

If  $e_{D \cup \mathcal{F}}(V \setminus (A \cup B), w_{r+1}) < \max(|A|, |B|)$ , then let  $z \in V \setminus (A \cup B)$  such that  $e_{D \cup \mathcal{F}}(z, B) = 0$ . Then OBreaker directs  $(z, w_{r+1})$ , updates  $\mathcal{F} := \mathcal{F} \cup \{(z, w_{r+1})\}$ , decreases  $t$  by one and repeats Step 2.

Otherwise,  $e_{D \cup \mathcal{F}}(V \setminus (A \cup B), w_{r+1}) = \max(|A|, |B|)$ . Then let  $\ell$  be the maximal index  $i \in [r]$  such that  $e_{D \cup \mathcal{F}}(V \setminus (A \cup B), w_i) < \max(|A|, |B|)$ . Let  $z \in V \setminus (A \cup B)$  such that  $(z, w_\ell) \notin D \cup \mathcal{F}$  and  $(z, w_{\ell+1}) \in D \cup \mathcal{F}$ . OBreaker directs  $(z, w_\ell)$ , updates  $\mathcal{F} := \mathcal{F} \cup \{(z, w_\ell)\}$ , decreases  $t$  by one and repeats Step 2.

We first show that OBreaker can follow the strategy. First note, by Property (R4) of a riskless digraph, that for all  $z \in V \setminus (A \cup B)$ , for all  $x \in A$  we have that  $(z, x) \notin D$ , and for all  $y \in B$  we have that  $(y, z) \notin D$ . Hence, under the assumption that  $x_A$  and  $y_B$  in Step 1 exist, OBreaker can follow the proposed strategy in Step 1.

Now, since  $D$  is a riskless digraph of rank  $r+1$  with  $|D| \leq (r+1)(b+1)$ , and since  $r \leq n/25 - 1$ , by Proposition 4.1 (i) we have that

$$\begin{aligned} |X_A| &= |\{z \in V \setminus (A \cup B) : e_D(A, z) = 0\}| > 2n/5 - 2 \text{ and} \\ |Y_B| &= |\{z \in V \setminus (A \cup B) : e_D(z, B) = 0\}| > 2n/5 - 2, \end{aligned}$$

before the first update in Step 1. On the other hand, in each iteration of Step 1,  $A$  or  $B$  increases by one vertex from  $V \setminus (A \cup B)$  (alternatingly). Since by (R4) there are no arcs inside  $V \setminus (A \cup B)$ , and since in Step 1, OBreaker directs all edges between the new vertices in  $A$  and those in  $B$ , the size of each of these two sets can increase by at most  $\sqrt{b} \leq \sqrt{n} \leq n/100$  for large enough  $n$ . Since  $X_A$  and  $Y_B$  consist of at least  $2n/5 - 2$  elements each before entering Step 1, the existence of  $x_A$  and  $y_B$  in each iteration of Step 1 follows.

For Step 2, the existence of  $z \in V \setminus (A \cup B)$  such that  $e_{D \cup \mathcal{F}}(z, B) = 0$  is guaranteed by the following: Consider a vertex  $z$  in  $Y_B$  before Step 1. Then  $e_D(z, B) = 0$  by definition, and  $e_D(z, V \setminus (A \cup B)) = 0$  by (R4). Now, if  $z$  is not added to  $A$  or  $B$  during Step 1, then  $e_D(z, B) = 0$  holds still after the update of  $B$ . Since in Step 1,  $\mathcal{F}$  contains only arcs between  $A$  and  $B$ , it follows, under the assumption that  $z \in V \setminus (A \cup B)$  after the update, that  $e_{D \cup \mathcal{F}}(z, B) = 0$  before entering Step 2. Since  $|Y_B| > 2n/5 - 2$  before entering Step 1, and since in Step 1 at most  $2\sqrt{n}$  vertices are moved from  $Y_B$  to  $A \cup B$ , it follows that at the beginning of Step 2, there are at least  $n/4$  vertices  $z$  in  $V \setminus (A \cup B)$  such that  $e_{D \cup \mathcal{F}}(z, B) = 0$ . Note that in Step 2 at most  $\max(|A|, |B|) - 1 \leq n/5 + \sqrt{n} < n/4$  of those vertices  $z \in V \setminus (A \cup B)$  with  $e_{D \cup \mathcal{F}}(z, B) = 0$  are used. The existence of  $\ell$  is always guaranteed since  $e_D(V \setminus (A \cup B), w_1) = 0$  by assumption, and since Step 2 is executed at most  $\max(|A|, |B|) - 1$  times. Note that by choice of  $x \in V \setminus (A \cup B)$ , OBreaker can always direct  $(x, w_{r+1})$  or  $(x, w_\ell)$  as required.

Finally, we prove that  $D' := D \cup \mathcal{F}$  is a riskless digraph of rank  $r+1$ , where  $\mathcal{F}$  is the set of all arcs that OBreaker directed in Step 1 and Step 2. In Step 1, the sets  $A$  and  $B$  are enlarged by one in each iteration. Since for each new element  $x_A$  (or  $y_B$  respectively) all arcs  $(x_A, y)$  for  $y \in B$  (or  $(x, y_B)$  for  $x \in A$  respectively) are directed by OBreaker (unless they are in  $D \cup \mathcal{F}$  already), the pair  $(A, B)$  is a  $UDB$  in  $D \cup \mathcal{F}$ . Since  $A$  and  $B$  are increased alternately (except for the first execution of Step 1 in case  $|B| = |A| + 1$ ), it follows that  $||A| - |B|| \leq 1$ . Since  $A_S$  and  $B_S$  are unchanged, Property (R1) follows.

Since  $A_S$  and  $B_S$  are untouched, there is nothing to prove for (R2.1). For (R2.2), note that, after the last update of Step 2, for all  $z \in A_0 \cup V \setminus (A \cup B)$ , the set  $\{i : (v_i, z) \in D \cup \mathcal{F}\}$  is

the same as  $\{i : (v_i, z) \in D\}$ . Now, for all  $z \in B_0 \cup V \setminus (A \cup B)$ , the arc  $(z, w_i)$  is directed by OBreaker for some  $1 \leq i < r + 1$  only if  $(z, w_{i+1}) \in D \cup \mathcal{F}$ . So (R2.3) follows as well. For (R3.1), note that in Step 1, all vertices  $z$  that are added to  $A_0$  fulfill  $e_D(A, z) = 0$ . Hence for all  $1 \leq i \leq r + 1$ ,  $e(v_i, A_0)$  does not (strictly) increase when proceeding from  $D$  to  $D \cup \mathcal{F}$ . Also,  $e(v_i, V \setminus (A \cup B))$  does not (strictly) increase, since all arcs of the form  $(v_i, z)$ , that are directed by OBreaker, fulfill  $z \in B$  after the update. In Step 2, only edges of the form  $(z, w_i)$  for  $z \in V \setminus (A \cup B)$  are directed, hence (R3.1) follows. For (R3.2), similar to (R3.1), the quantity  $e(B_0, w_i)$  does not increase in Step 1, for all  $1 \leq i \leq r + 1$ , since all vertices  $z \in V \setminus (A \cup B)$  added to  $B_0$  fulfill  $e_D(z, B) = 0$ . In Step 2, no vertices are added to  $B_0$ , so the quantity  $e(B_0, w_i)$  stays unchanged for all  $1 \leq i \leq r + 1$ . Now for  $1 \leq i \leq r + 1$ , the quantity  $e(V \setminus (A \cup B), w_i)$  only increases in Step 2, and only if  $e(V \setminus (A \cup B), w_i) < \max(|A|, |B|)$  by the strategy description. Therefore, (R3.2) follows. Finally, Property (R4) follows since OBreaker updates  $A$  and  $B$  accordingly in Step 1, and since in Step 2,  $\delta$  only directs arcs of the form  $(x, w_i)$  for  $x \in V \setminus (A \cup B)$  and  $w_i \in B_S$ .  $\square$

## 5 Strict rules - Stage II

*Proof of Lemma 3.7.* By assumption,  $D$  is a riskless digraph of rank  $r$ . Let  $A = A_S \cup A_0$  and  $B = B_S \cup B_0$  be given according to Definition 3.1. We claim that  $D$  is protected with  $UDB(A, B)$  with partitions  $A = A_D \cup A_{AD} \cup A_S \cup A_0$  and  $B = B_D \cup B_{AD} \cup B_S \cup B_0$ , where  $A_D = A_{AD} = B_D = B_{AD} = \emptyset$ , and  $k_1 = \ell_1 = 0$ ,  $k_2 = \ell_2 = r$ .

For Property (P1), let  $a_0 := |A_0|$  and note that by Property (R1),  $|a_0 - |B_0|| \leq 1$ . By assumption on  $|D|$  and by Property (R4),

$$r(b + 1) = |D| = e_D(A, B) + e_D(A_S, V \setminus B) + e_D(V \setminus A, B_S). \quad (1)$$

Now,  $e_D(A, B) = (r + a_0)(r + |B_0|) \leq (r + a_0 + 1)^2$ ; whereas

$$\begin{aligned} e_D(A_S, V \setminus B) &= e_D(A_S, A_S) + e_D(A_S, A_0) + e_D(A_S, V \setminus (A \cup B)) \\ &\leq \binom{r}{2} + \frac{r(r+1)}{2} + r(a_0 + 1 + r) \\ &\leq r^2 + r(a_0 + 1 + r) \end{aligned}$$

where the first inequality follows from Property (R1) and (R3.1). Similarly, by Property (R1) and (R3.2),  $e_D(V \setminus A, B_S) \leq r^2 + r(a_0 + 1 + r)$ . Thus, (1) yields

$$r(b + 1) \leq (r + a_0 + 1)^2 + 4r^2 + 2r(a_0 + 1).$$

Standard, but slightly tedious calculations show that this implies that  $a_0 + 1 \geq n/10 + 3$  for  $r = \lfloor n/25 \rfloor$ ,  $b \geq 19n/20$  and  $n$  large enough. This then implies that  $|B_0| \geq a_0 - 1 \geq n/10 + 1$ .

There is nothing to prove for Property (P2) and (P3) since  $A_D = A_{AD} = B_D = B_{AD} = \emptyset$ . For Property (P4), note that  $A_{AD} = B_{AD} = \emptyset$  and the enumerations  $A_S = \{v_1, \dots, v_r\}$  and  $B_S = \{w_1, \dots, w_r\}$  given by Property (R2) fulfil (P4.1)-(P4.3), with  $k_1 = \ell_1 = 0$  and  $k_2 = \ell_2 = r$ . Property (P5.1) and (P5.2) follow from (R3.1) and (R3.2) respectively. Finally, (P6) follows from (R4).  $\square$

*Proof of Lemma 3.8.* Let  $A = A_D \cup A_{AD} \cup A_S \cup A_0$  and  $B = B_D \cup B_{AD} \cup B_S \cup B_0$  be given by Definition 3.5, and let  $A_{AD} = \{v_1, \dots, v_{k_1}\}$ ,  $A_S = \{v_{k_1+1}, \dots, v_{k_1+k_2}\}$  and  $B_S = \{w_1, \dots, w_{\ell_2}\}$ ,  $B_{AD} = \{w_{\ell_2+1}, \dots, w_{\ell_1+\ell_2}\}$  as given by Property (P4). Let  $e = (v, w) \in \mathcal{A}(D(V \setminus B))$  be given by the lemma. For notational reasons we divide into two cases.

**Case 1:**  $v \in A_{AD} \cup A_S$ . Then  $v = v_\ell$  for some  $1 \leq \ell \leq k_1 + k_2$ . Note that the only properties that may not be fulfilled anymore in  $D+e$  are (P4.2) and (P5.1). Let  $\{f_1, \dots, f_t\} = (\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3) \cap \mathcal{A}(D)$ , where

$$\begin{aligned}\mathcal{F}_1 &:= \{(v_i, w) : 1 \leq i \leq \ell - 1\}, \\ \mathcal{F}_2 &:= \{(v_1, z) : z \in A_0\}, \\ \mathcal{F}_3 &:= \{(v_{k_1+1}, z) : z \in V \setminus (A \cup B)\},\end{aligned}$$

where we use the convention that  $\mathcal{F}_3 = \emptyset$  if  $A_S = \emptyset$ . To show that this choice of arcs is suitable for the lemma, we now follow the structure of the proof of Lemma 3.3. That is, we prove that  $f_i \neq \overleftarrow{e}$  for all  $1 \leq i \leq t$ , that  $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \subseteq D \cup \mathcal{A}(D)$ , that  $t \leq b$ , and finally we deduce that  $D' = D \cup \{e, f_1, \dots, f_t\}$  is protected.

For the first part,  $\overleftarrow{e} = (w, v) \notin \mathcal{F}_1$  since  $v \neq w$ , and  $\overleftarrow{e} = (w, v) \notin \mathcal{F}_2 \cup \mathcal{F}_3$  since  $v \in A_{AD} \cup A_S$  by assumption. To see that  $\mathcal{F}_1 \subseteq D \cup \mathcal{A}(D)$ , assume that  $(w, v_i) \in D$  for some  $1 \leq i \leq \ell - 1$ . Then  $w = v_j \in A_{AD} \cup A_S$  for some  $j < i < \ell$ , by Property (P6), (P4.1) and since  $w \notin A_D$ . By Property (P4.1) again, we conclude  $(w, v) = (v_j, v_\ell) \in D$ , a contradiction to  $e \in \mathcal{A}(D(V \setminus B))$ . Thus,  $\mathcal{F}_1 \subseteq D \cup \mathcal{A}(D)$ . Now,  $\mathcal{F}_2 \cup \mathcal{F}_3 \subseteq D \cup \mathcal{A}(D)$  since every arc of the form  $(z, v_i)$  in  $D$  satisfies  $z \in A \setminus A_0$ , by Property (P6). To see that  $t \leq b$  note that

$$t \leq |\mathcal{F}_1| + |\mathcal{F}_2| + |\mathcal{F}_3| \leq \ell + |A_0| + |V \setminus (A \cup B)| \leq |V \setminus B| \leq b,$$

since by Property (P1),  $|B| \geq n/10 + 1 \geq n - b$ .

To check that  $D'$  is protected, consider the partition  $A = A'_D \cup A'_{AD} \cup A'_S \cup A_0$  where  $A'_D := A_D \cup \{v_1\}$ ,  $A'_{AD} := \{v_2, \dots, v_{k_1+1}\}$  (or  $A'_{AD} := \{v_2, \dots, v_{k_1}\}$  if  $k_2 = 0$ ), and  $A'_S := \{v_{k_1+2}, \dots, v_{k_1+k_2}\}$ . Clearly,  $(A, B)$  is still a  $UDB$  in  $D'$  with  $|A|, |B| \geq n/10 + 1$ , and Property (P1) holds since  $|A'_D \cup A_0| = |A_D \cup A_0| + 1$ .

For Property (P2),  $B_D$  did not change; and  $D'(A'_D)$  is a transitive tournament, since  $D(A_D) \subseteq D'(A'_D)$  is such, and since  $(z, v_1) \in D \subseteq D'$  for every  $z \in A_D$  by the  $UDB$ -property for  $A_D$ . To see that  $(A'_D, V \setminus A'_D)$  forms a  $UDB$  we need to observe that  $(v_1, z) \in D'$  for every  $z \in V \setminus A'_D$ . If  $v_1 \in A_{AD}$ , then this follows by Property (P3) and (P4.1) for  $D$ , and since  $\mathcal{F}_2 \subseteq D'$ . If  $v_1 \in A_S$  (and thus  $k_1 = |A_{AD}| = 0$ ), then this follows since  $(A, B)$  is a  $UDB$  in  $D$ , by Property (P4.1) for  $D$ , and since  $\mathcal{F}_2 \cup \mathcal{F}_3 \subseteq D'$ .

To see that Property (P3) holds in  $D'$ , observe first that  $A'_{AD} \setminus \{v_{k_1+1}\} \subseteq A_{AD}$ . Moreover,  $(v_{k_1+1}, z) \in D'$  for every  $z \in V \setminus A$  since  $(A, B)$  is a  $UDB$  in  $D$  and since  $\mathcal{F}_3 \subseteq D'$ . For Property (P4), it obviously holds that  $|A'_S| < k_2 \leq n/25$  and that  $|B_S| \leq n/25$ . Property (P4.1) and (P4.3) follow trivially, Property (P4.2) follows from Property (P4.2) for  $D$  and since  $\mathcal{F}_1 \subseteq D'$ .

For (P5.1), observe that, since we made an index shift (from  $A_S$  to  $A'_S$ ), we have to prove that  $e_{D'}(v_{(k_1+1)+i}, A_0) \leq n/25 + 1 - i$  for every  $1 \leq i < k_2$ . First, let  $1 \leq i < k_2$  be such that  $(k_1 + 1) + i \leq \ell$ . Then only arcs from  $D(v_{k_1+1+i}, A_0) \cup \mathcal{F}_1 \cup \{e\}$  contribute to  $D'(v_{k_1+1+i}, A_0)$ . Therefore,  $e_{D'}(v_{k_1+1+i}, A_0) \leq e_D(v_{k_1+1+i}, A_0) + 1 \leq (n/25 + 1 - (1+i)) + 1$ . Now let  $k_1 + 1 + i > \ell$ . Then  $D'(v_{k_1+1+i}, A_0) = D(v_{k_1+1+i}, A_0)$ , and therefore,  $e_{D'}(v_{k_1+1+i}, A_0) \leq n/25 + 1 - i$ .

There is nothing to prove for Property (P5.2). Finally, Property (P6) follows since  $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \subseteq D'(A \setminus A_0, V \setminus B)$  and therefore,

$$\begin{aligned} D' &= D \cup \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \\ &= D(A, B) \cup D(A \setminus A_0, V \setminus B) \cup D(V \setminus A, B \setminus B_0) \cup \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \\ &= D'(A, B) \cup D'(A \setminus A_0, V \setminus B) \cup D'(V \setminus A, B \setminus B_0). \end{aligned}$$

**Case 2:**  $v \notin A_{AD} \cup A_S$ . By Property (P2) and since  $e = (v, w) \in \mathcal{A}(D(V \setminus B))$ , we may assume that  $v \notin A_D$ , i.e.  $v \in A_0 \cup V \setminus (A \cup B)$ . We now want to incorporate  $v$  into the tournament  $A_{AD} \cup A_S$ . Set

$$\ell := \begin{cases} \min(i : (v_i, v) \notin D) & \text{if minimum exists} \\ k_1 + k_2 + 1 & \text{otherwise,} \end{cases}$$

and

$$v'_i := \begin{cases} v_i & 1 \leq i \leq \ell - 1 \\ v & i = \ell \\ v_{i-1} & \ell + 1 \leq i \leq k_1 + k_2 + 1. \end{cases}$$

Consider the following families of arcs.

$$\begin{aligned} \mathcal{F}_1 &:= \{(v'_i, w) : 1 \leq i \leq \ell - 1\} \\ \mathcal{F}_2 &:= \{(v'_\ell, v'_i) : \ell + 1 \leq i \leq k_1 + k_2 + 1\} \\ \mathcal{F}_3 &:= \{(v'_\ell, z) : z \in V \setminus (A \cup B) \cup A_0 \text{ and } (v'_{\ell+1}, z) \in D\} \\ \mathcal{F}_4 &:= \begin{cases} \{(v'_1, z) : z \in A_0 \text{ and } (v'_{\ell+1}, z) \notin D\} & \text{if } v \in A_0 \\ \{(v'_\ell, y) : y \in B\} & \text{if } v \in V \setminus (A \cup B) \end{cases} \\ \mathcal{F}_5 &:= \{(v'_{k_1+1}, z) : z \in V \setminus (A \cup B), z \neq v \text{ and } (v'_{\ell+1}, z) \notin D\}, \end{aligned}$$

where we use the convention that if  $\ell = k_1 + k_2 + 1$  (and thus  $v'_{\ell+1}$  does not exist) then we take  $\mathcal{F}_3 = \emptyset$ ,  $\mathcal{F}_5 := \{(v'_{k_1+1}, z) : z \in V \setminus (A \cup B), z \neq v\}$ , and  $\mathcal{F}_4 := \{(v'_1, z) : z \in A_0\}$  when  $v \in A_0$ . We illustrate the arcs in  $\mathcal{F}_1 \cup \dots \cup \mathcal{F}_5$  in Figure 7.

We choose  $\{f_1, \dots, f_t\}$  to be  $\{f_1, \dots, f_t\} = (\mathcal{F}_1 \cup \mathcal{F}_2 \cup \dots \cup \mathcal{F}_5) \cap \mathcal{A}(D)$ . We proceed as before and show that  $f_i \neq \overleftarrow{e}$  for all  $1 \leq i \leq t$ , that  $\mathcal{F}_1 \cup \dots \cup \mathcal{F}_5 \subseteq D \cup \mathcal{A}(D)$ , that  $t \leq b$ , and finally we deduce that  $D' = D \cup \{e, f_1, \dots, f_t\}$  is protected.

For the first part,  $\overleftarrow{e} = (w, v) \notin \mathcal{F}_1$  since  $v \neq w$ . Similarly,  $\overleftarrow{e} = (w, v) \notin \mathcal{F}_2 \cup \mathcal{F}_3$  since  $v'_\ell = v \neq w$ . For the same reason,  $\overleftarrow{e} \notin \mathcal{F}_4$  in the case when  $v \in V \setminus (A \cup B)$ . In the case when  $v \in A_0$ , assume that  $(w, v) = (v'_1, z)$  for some  $z \in A_0$ . Then  $(v'_1, v) = (w, v) \in \mathcal{A}(D)$ , by assumption on  $e$ , and  $v'_1 = v_1$ . That is,  $(v'_1, v) \notin D$ , and  $\ell = 1$  by definition of  $\ell$ . But then  $w = v'_1 = v'_\ell = v$ , by definition of  $v'_i$ , a contradiction. Also,  $\overleftarrow{e} \notin \mathcal{F}_5$  by definition of  $\mathcal{F}_5$ . So,  $f_i \neq \overleftarrow{e}$  for all  $1 \leq i \leq t$ .

To see that  $\mathcal{F}_1 \subseteq D \cup \mathcal{A}(D)$ , assume that  $(w, v'_i) = (w, v_i) \in D$  for some  $1 \leq i \leq \ell - 1$ . Then  $w = v_j \in A_{AD} \cup A_S$  for some  $j < i < \ell$ , by Property (P6), (P4.1) and since  $w \notin A_D$ . By definition of  $\ell$  we conclude  $(w, v) = (v_j, v) \in D$ , a contradiction to  $e \in \mathcal{A}(D(V \setminus B))$ . Thus,



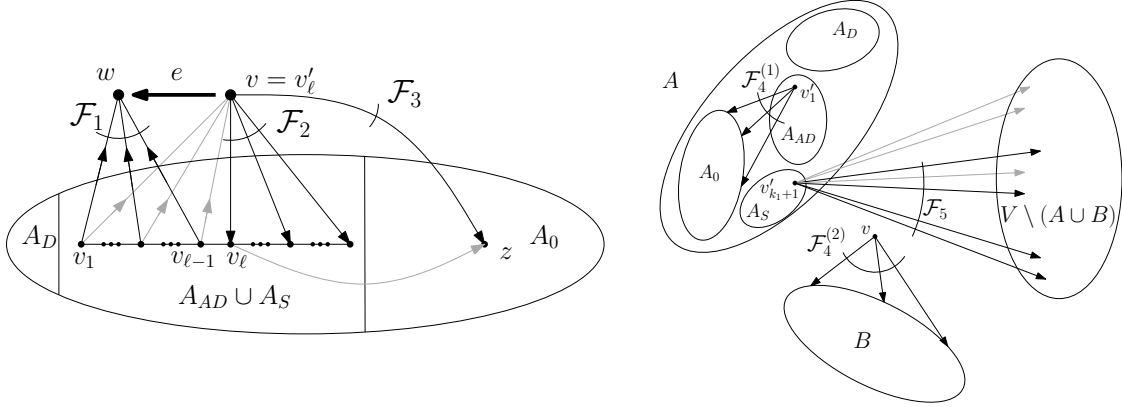


Figure 7: The arcs OBreaker directs in Stage II. Grey arcs are in  $D$  already before OMaker's move. In case  $v \in V \setminus (A \cup B)$ , the family  $\mathcal{F}_2$  includes  $v$  into the tournament in  $A$ ,  $\mathcal{F}_1$  and  $\mathcal{F}_3$  restore (P4.2). The family  $\mathcal{F}_4^{(1)}$  adds  $v'_1$  to  $A_D$  in case  $v \in A_0$ ; otherwise (if  $v \in V \setminus (A \cup B)$ ) the family  $\mathcal{F}_4^{(2)}$  adds  $v$  to  $A$ . Finally,  $\mathcal{F}_5$  adds  $v'_{k_1+1}$  to  $A_{AD}$ .

$\mathcal{F}_1 \subseteq D \cup \mathcal{A}(D)$ . Now,  $\mathcal{F}_2 \subseteq D \cup \mathcal{A}(D)$  by Property (P4.2) and the definition of  $\ell$ . Also,  $\mathcal{F}_3 \cup \mathcal{F}_4 \cup \mathcal{F}_5 \subseteq D \cup \mathcal{A}(D)$  since every arc of the form  $(w', v'_i)$  in  $D$  satisfies  $w' \in A \setminus A_0$ , by Property (P6) and since  $v'_i \in A \cup V \setminus (A \cup B)$ .

To bound the number  $t$  by the bias  $b$  note that in case  $v \in A_0$ ,

$$\begin{aligned} t &\leq |\mathcal{F}_1 \cup \mathcal{F}_2| + |\mathcal{F}_3 \cup \mathcal{F}_4 \cup \mathcal{F}_5| \\ &\leq k_1 + k_2 + |A_0 \cup V \setminus (A \cup B)| \\ &= |A_{AD} \cup A_S| + |A_0 \cup V \setminus (A \cup B)| \leq |V \setminus B| < \frac{9n}{10} < b, \end{aligned}$$

since by Property (P1) we have  $|B| \geq n/10 + 1$ . When  $v \in V \setminus (A \cup B)$ , then we estimate

$$\begin{aligned} t &\leq |\mathcal{F}_1 \cup \mathcal{F}_2| + |\mathcal{F}_3 \cup \mathcal{F}_5| + |\mathcal{F}_4| \\ &\leq |A_{AD} \cup A_S| + (|V \setminus (A \cup B)| + e_D(v'_{\ell+1}, A_0)) + |B| \\ &= |V \setminus (A_0 \cup A_D)| + e_D(v'_{\ell+1}, A_0). \end{aligned}$$

Since  $v \in V \setminus (A \cup B)$ , we have that  $\ell \geq k_1 + 1$ , by Property (P3) and definition of  $\ell$ . Since  $v'_{\ell+1} = v_\ell$ , it follows that

$$t \leq |V \setminus (A_0 \cup A_D)| + e_D(v'_{\ell+1}, A_0) \leq \frac{9n}{10} + \frac{n}{25} < b,$$

by Property (P1), (P5.1) and choice of  $b$ .

Finally, we show that  $D' = D \cup \{e, f_1, \dots, f_t\}$  is a protected digraph. Set

$$A'_D := \begin{cases} A_D \cup \{v'_1\} & \text{if } v \in A_0, \\ A_D & \text{if } v \in V \setminus (A \cup B) \end{cases} \quad A'_{AD} := \begin{cases} \{v'_2, \dots, v'_{k_1+1}\} & \text{if } v \in A_0 \\ \{v'_1, \dots, v'_{k_1+1}\} & \text{if } v \in V \setminus (A \cup B) \end{cases}$$

$$A'_S := \{v'_{k_1+2}, \dots, v'_{k_1+k_2+1}\} \quad A'_0 := A_0 \setminus \{v\} \quad A' := A \cup \{v\}.$$

Moreover, let  $B' = B$  with the same partition as for  $B$ . Then  $(A', B')$  is a  $UDB$  in  $D'$ , since  $(A, B)$  is a  $UDB$  in  $D \subseteq D'$  and since in case  $v \in V \setminus (A \cup B)$  we have  $\mathcal{F}_4 \subseteq D'$ . For Property (P1),  $|B_D \cup B_0| \geq n/10$  and  $|A'|, |B'| \geq n/10 + 1$  obviously hold. Now, observe that  $|A'_D| = |A_D| + 1$  and  $|A'_0| = |A_0| - 1$  in case  $v \in A_0$ , while  $A_D = A'_D$  and  $A_0 = A'_0$  in case  $v \in V \setminus (A \cup B)$ . Thus  $|A'_D \cup A'_0| = |A_D \cup A_0| \geq n/10$ .

For Property (P2), there is nothing to prove when  $v \in V \setminus (A \cup B)$ , since then the sets of dead vertices do not change. So, let  $v \in A_0$ . Again  $B_D$  does not change.  $D'(A'_D)$  is a transitive tournament, since  $D(A_D) \subseteq D'(A'_D)$  is such, and since  $(z, v'_1) \in D \subseteq D'$  for every  $z \in A_D$  by the  $UDB$ -property for  $A_D$ . To see that  $(A'_D, V \setminus A'_D)$  forms a  $UDB$  we need to observe that  $(v'_1, z) \in D'$  for every  $z \in V \setminus A'_D$ .

Assume first that  $v'_1 = v_1 \in A_{AD} \cup A_S$ . Then  $(v'_1, z) \in D \subseteq D'$  for every  $z \in B$  since  $(A, B)$  is a  $UDB$ ; and for every  $z \in (A_{AD} \cup A_S) \setminus \{v'_1\}$  by Property (P4.1). For every  $z \in A_0$ , if  $(v'_{\ell+1}, z) \notin D$  (or  $\ell = k_1 + k_2 + 1$  where  $v'_{\ell+1}$  does not exist), then  $(v'_1, z) \in \mathcal{F}_4 \subseteq D'$ ; and if  $(v'_{\ell+1}, z) \in D$  then  $(v'_1, z) \in D \subseteq D'$  by Property (P4.2). For  $z \in V \setminus (A \cup B)$ , if  $v'_1 \in A_{AD}$  then  $(v'_1, z) \in D \subseteq D'$  by Property (P3) for  $D$ ; if  $v'_1 \in A_S$  (and thus  $k_1 = |A_{AD}| = 0$ ) and if  $(v'_{\ell+1}, z) \notin D$  (or  $\ell = k_1 + k_2 + 1$  where  $v'_{\ell+1}$  does not exist), then  $(v'_1, z) \in \mathcal{F}_5 \subseteq D'$ ; and if  $v'_1 \in A_S$  and  $(v'_{\ell+1}, z) \in D$ , then  $(v'_1, z) \in D \subseteq D'$  by Property (P4.2).

Now, assume that  $v'_1 = v \in A_0$  and thus  $\ell = 1$ . If  $v'_2 = v'_{\ell+1} \in A_{AD}$ , then  $(v'_2, z) \in D$  for every  $z \in V \setminus (A \cup B)$  by Property (P3). Therefore, for every  $z \in (A_0 \cup V \setminus (A \cup B)) \setminus \{v\}$  we have that  $(v'_1, z) \in \mathcal{F}_3 \cup \mathcal{F}_4 \subseteq D'$ . For every  $z \in A_{AD} \cup A_S$ , we have that  $(v'_1, z) \in \mathcal{F}_2 \subseteq D'$ . Furthermore,  $(v'_1, z) \in D \subseteq D'$  for every  $z \in B$  since  $v'_1 \in A_0$  and  $(A, B)$  is a  $UDB$  in  $D$ . If  $v'_2 \in A_S$  (or  $v'_2$  does not exist) and therefore  $k_1 = 0$ , then similarly  $(v'_1, z) \in \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4 \cup \mathcal{F}_5 \cup D \subseteq D'$  for all  $z \in V \setminus A'_D$ .

For Property (P3), observe that the statement for  $B_{AD}$  does not change. To see that  $(A'_{AD}, V \setminus A')$  forms a  $UDB$  we need to observe that  $(v'_{k_1+1}, z) \in D'$  for every  $z \in V \setminus A'$ . If  $z \in B'$ , then this is clear, since  $(A', B')$  forms a  $UDB$  as we showed already above. Let now  $z \in V \setminus (A' \cup B') = V \setminus (A \cup B \cup \{v\})$ . If  $\ell \leq k_1$ , then  $v'_{k_1+1} = v_{k_1} \in A_{AD}$ . Therefore,  $(v'_{k_1+1}, z) \in D \subseteq D'$  by Property (P3) for  $D$ . If  $\ell \geq k_1 + 1$ , then  $(v'_{k_1+1}, z) \in \mathcal{F}_3 \cup \mathcal{F}_5 \cup D \subseteq D'$  (where we use Property (P4.2) which says that if  $(v'_{\ell+1}, z) = (v_\ell, z) \in D$  then  $(v_{k_1+1}, z) \in D$ ).

For Property (P4), note that  $|A'_S| = |A_S| = k_2 \leq n/25$ . For (P4.1) observe that the statement for  $\{w_1, \dots, w_{\ell_1+\ell_2}\}$  does not change. To see that  $(v'_2, \dots, v'_{k_1+k_2+1})$  or  $(v'_1, \dots, v'_{k_1+k_2+1})$  induces a transitive tournament, note first that the vertex set without  $v'_\ell$  induces a transitive tournament in  $D \subseteq D'$ . We have  $(v'_i, v'_\ell) = (v_i, v) \in D \subseteq D'$  for every  $i \leq \ell - 1$ , by definition of  $\ell$ . Moreover,  $(v'_\ell, v'_i) \in D'$  for every  $i \geq \ell + 1$ , since  $\mathcal{F}_2 \subseteq D'$ .

For (P4.2), let  $z \in A'_0 \cup V \setminus (A' \cup B')$ . We show that  $\{i : (v'_i, z) \in D\}$  is a down set of  $[k_1 + k_2 + 1]$ . Note that this then implies (P4.2) also when  $v \in A_0$  and thus  $A'_{AD} \cup A'_S = \{v'_2, \dots, v'_{k_1+k_2+1}\}$ . Since  $z \in A'_0 \cup V \setminus (A' \cup B')$ , only arcs from  $D \cup \{e\} \cup \mathcal{F}_1 \cup \mathcal{F}_3 \cup \mathcal{F}_4 \cup \mathcal{F}_5$  contribute to the set  $\{i : (v'_i, z) \in D'\}$ . Note that  $\{i : (v_i, z) \in D\}$  is a down set of  $[k_1 + k_2]$  and the relative order of the  $v'_i$  for  $i \neq \ell$  does not change. So, if  $z \neq w$ , then the arcs from  $\mathcal{F}_3$  reestablish the down-set property for  $D'$ . If  $z = w$ , then the arcs from  $\mathcal{F}_1$  reestablish the down-set property for  $D'$ . Now,  $\mathcal{F}_4$  contributes at most the element  $\{1\}$  to  $\{i : (v'_i, z) \in D'\}$  which is of no harm. The family  $\mathcal{F}_5$  may contribute the element  $\{k_1 + 1\}$  to  $\{i : (v'_i, z) \in D'\}$  for some  $z \in V \setminus (A \cup B)$ . However, by Property (P3),  $(v_i, z) \in D$  for all  $1 \leq i \leq k_1$ , so  $\mathcal{F}_5$  certainly does not destroy the down-set property.

There is nothing to prove for Property (P4.3), since  $B$  and  $\{w_1, \dots, w_{\ell_1+\ell_2}\}$  are unchanged.

For (P5.1), observe that, since we make an index shift (from  $A_S$  to  $A'_S$ ), we have to prove that  $e_{D'}(v'_{(k_1+1)+i}, A'_0) \leq n/25 + 1 - i$  for every  $1 \leq i \leq k_2$ . First, let  $1 \leq i \leq k_2$  be such that  $(k_1 + 1) + i < \ell$ . Then only arcs from  $D(v_{k_1+1+i}, A_0) \cup \mathcal{F}_1$  contribute to  $D'(v'_{k_1+1+i}, A'_0)$ . Therefore,  $e_{D'}(v'_{k_1+1+i}, A'_0) \leq e_D(v_{k_1+1+i}, A_0) + 1 \leq (n/25 + 1 - (1 + i)) + 1$ .

Now, let  $(k_1 + 1) + i = \ell$ . Then  $e_D(v'_\ell, A_0) = e_D(v, A_0) = 0$  since  $v \in A_0 \cup V \setminus (A \cup B)$  and by Property (P6) for  $D$ . So, only  $\mathcal{F}_3 \cup \{e\}$  contributes to  $D'(v'_\ell, A'_0)$ . Therefore,  $e_{D'}(v'_\ell, A'_0) \leq e_D(v'_{\ell+1}, A_0) + 1 = e_D(v_\ell, A_0) + 1 = e_D(v_{k_1+1+i}, A_0) + 1 \leq n/25 + 1 - i$ . Finally, let  $k_1 + 1 + i > \ell$ . Then  $v'_{k_1+1+i} = v_{k_1+i}$  and only arcs from  $D(v_{k_1+i}, A_0)$  contribute to  $D'(v'_{k_1+1+i}, A'_0)$ . This again proves  $e_{D'}(v'_{k_1+1+i}, A'_0) \leq n/25 + 1 - i$ .

There is nothing to prove for Property (P5.2).

For (P6) note that it is enough to prove that

$$D' \subseteq D'(A', B') \cup D'(A' \setminus A'_0, V) \cup D'(V, B' \setminus B'_0).$$

This indeed holds, since

$$\begin{aligned} D(A, B) &\subseteq D'(A', B'), \\ D(A \setminus A_0, V \setminus B) \cup \mathcal{F}_1 \cup \dots \cup \mathcal{F}_5 &\subseteq D'(A' \setminus A'_0, V), \\ D(V \setminus A, B \setminus B_0) &\subseteq D'(V, B' \setminus B'_0). \end{aligned} \quad \square$$

*Proof of Lemma 3.9.* Assume first that  $A_{AD} \neq \emptyset$ . If  $(v_1, y) \in D$  for all  $y \in A_0$ , where  $v_1$  is the top vertex in the tournament on  $A_{AD}$  as before, then we may set  $A_D := A_D \cup \{v_1\}$  and  $A_{AD} := A_{AD} \setminus \{v_1\}$  and reapply the Lemma. Otherwise, there exists a vertex  $y \in A_0$  such that the pair  $\{v_1, y\}$  is not directed. Then the arc  $f = (v_1, y)$  obviously satisfies that  $D + f$  is protected.

So assume from now on that  $A_{AD} = \emptyset$ . If  $A_S \neq \emptyset$  let  $v_1$  be the top element in the transitive tournament  $D(A_S)$ . We similarly may assume that there exists a vertex  $y \in V \setminus (A \cup B)$  such that the pair  $\{v_1, y\}$  is not directed, for otherwise we reapply the Lemma with  $A_{AD} := \{v_1\}$  and  $A_S := A_S \setminus \{v_1\}$ . Then the arc  $f = (v_1, y)$  obviously satisfies that  $D + f$  is protected.

So assume from now on that  $A_{AD} = A_S = \emptyset$ . If  $A_0 \neq \emptyset$  and  $V \setminus (A \cup B) \neq \emptyset$ , then let  $f = (x, y)$  for some  $x \in A_0$  and some  $y \in V \setminus (A \cup B)$ . Now set  $A_S := \{x\}$  and  $A_0 := A_0 \setminus \{x\}$ . It is easy to see that all the properties (P2)-(P6) hold for  $D + f$  after the update. For (P1), note that by assumption,  $|A_D \cup A_0| + 1 = |A| \geq n/10 + 1$  after the update. Hence,  $|A_D \cup A_0| \geq n/10$ . If  $|A_0| \geq 2$  and  $V \setminus (A \cup B) = \emptyset$ , then let  $f = (x, y)$  for some distinct vertices  $x, y \in A_0$ , and set  $A_{AD} := \{x\}$  and  $A_0 := A_0 \setminus \{x\}$ . The properties follow as in the previous case. If  $A_0 = \{x\}$  is a singleton and  $V \setminus (A \cup B) = \emptyset$ , set  $A_D := A_D \cup A_0$  and  $A_0 = \emptyset$  and reapply the lemma. So we may assume that  $A_{AD} = A_S = A_0 = \emptyset$ , that is,  $A = A_D$  and thus  $D(A)$  is a transitive tournament and  $(A, V \setminus A)$  is a *UDB*.

Now, by a similar analysis, we can either pick a suitable arc  $f$  in  $V \setminus A$  or deduce that  $B = B_D$ , that is,  $D(B)$  forms a transitive tournament and  $(V \setminus B, B)$  is a *UDB*. By assumption,  $D$  is not a transitive tournament on  $K_n$ , hence there must be an undirected pair  $\{x, y\}$ . Since  $A = A_D$  and  $B = B_D$ , both  $x, y$  must lie in  $V \setminus (A \cup B)$ . We claim that  $f = (x, y)$  is suitable: Set  $A_S := \{x\}$  and update  $A := A \cup \{x\}$ . Note that since  $(V \setminus B, B)$  is a *UDB*, all the arcs  $(x, z)$  for  $z \in B$  are elements of  $D$ . It is easy to see that the properties (P1)-(P6) hold for  $D + f$  after the update.  $\square$

## 6 Concluding remarks and open problems

In Theorem 1.2, we provide a winning strategy for OBReaker in the monotone Oriented-cycle game when  $b \geq 5n/6 + 2$  and thus prove that  $t(n, \mathcal{C}) \leq 5n/6 + 1$ . On the other hand [6],  $n/2 - 2$  is the best known lower bound on  $t(n, \mathcal{C})$ . In the proof of this result, OMaker first builds a long directed path (of length  $n - 1$ ) in at most  $n - 1$  rounds. When  $b \leq n/2 - 2$ , OBReaker cannot have directed all “backward” edges of this path, so OMaker can direct one of those and close a cycle. From the perspective of OBReaker, it is indeed most harmful if OMaker builds a long path. As we have seen in the motivation of  $\alpha$ -structures, OMaker can create  $\ell - 1$  immediate threats in the  $\ell^{\text{th}}$  round with such a strategy. On the other hand, OBReaker can ensure that  $\sigma$  needs to direct at most  $\ell - 1$  edges to answer every immediate threat, even if OMaker plays another strategy than building a long path (cf. Lemma 2.8). Despite these seemingly matching strategies, a significant gap between lower and upper bound remains. Let us describe briefly why  $b \geq n/\sqrt{2} - o(n)$  is necessary for our strategy to be a winning strategy for OBReaker.

Recall that our strategy for OBReaker included to build a  $UDB(A, B)$  such that both parts have size at least  $n - b$ . Suppose OBReaker succeeds to build a  $UDB(A, B)$  of size  $n - b - 1$  in some round  $r$  (in Stage I). Observe that then  $r \geq (n - b - 1)^2 / (b + 1)$ . Suppose that OMaker only directs edges in  $V \setminus B$  in the first  $r$  rounds, so that  $k$ , the size of the  $\alpha$ -structure in  $V \setminus B$ , increases in each round; whereas  $\ell$ , the size of the  $\alpha$ -structure in  $V \setminus A$ , is zero. Assume further that OBReaker only increases one of the values  $|A| - k = |A| - r$  and  $|B| - \ell = |B|$  (in order to decrease the number of edges  $\sigma$  has to direct in the next round). Without loss of generality, let this be  $|A| - k$ . In order to follow procedure  $\alpha$  and to increase  $|A| - k$  (by adding two vertices to  $A$ ), OBReaker needs to direct at least  $k + 2|B| \geq (n - b - 1)^2 / (b + 1) + 2(n - b - 1)$  edges in round  $r + 1$ . But this is only possible if  $b \geq n/\sqrt{2} - o(n)$ . We conjecture that the correct threshold is asymptotically at least  $n/\sqrt{2}$ .

**Conjecture 6.1.** *For  $b \leq n/\sqrt{2} - o(n)$  OMaker has a strategy to close a directed cycle in the monotone  $b$ -biased orientation game.*

It is of course desirable to determine the threshold bias for the monotone Oriented-cycle game  $t(n, \mathcal{C})$  exactly. Concerning the strict rules, we wonder whether  $t^+(n, \mathcal{C})$  and  $t^-(n, \mathcal{C})$  are (asymptotically) equal.

Here are two natural variants of the Oriented-cycle game.

### Playing on random graphs.

Suppose we replace the edge set of the complete graph  $K_n$  by the edges of a random graph  $G \sim \mathcal{G}_{n,p}$ , for some  $p = p(n)$ . That is, OMaker and OBReaker only direct edges of  $G$ . OMaker wins if the final digraph (with underlying edge set of  $G$ ) contains a directed cycle; otherwise, OBReaker wins. During the Berlin-Poznań Seminar in 2013, Łuczak asked how the threshold bias behaves in this variant of the game.

**Problem 6.2.** *Given  $0 < p = p(n) < 1$ . What is the largest bias  $b = b(n, p)$  such that OMaker asymptotically almost surely has a strategy to create an oriented cycle in the  $b$ -biased orientation game played on the edge set of  $G \sim \mathcal{G}_{n,p}$ , under the strict rules and under the monotone rules, respectively?*

Note that for any graph  $G$  with maximum degree  $\Delta$ , a modified version of the trivial strategy shows that OBreaker can win the Oriented-cycle game played on the edge set of  $G$  when  $b \geq \Delta - 1$ . Therefore,  $b(n, p) \leq (1 + o(1))np$ , provided  $p$  is large enough. As we show in this paper, this is not tight for  $p = 1$ . Indeed, we believe that for smaller values of  $p$  the trivial upper bound of  $np$  is not tight as well. We want to remark that in general our strategy does not directly yield an improvement on this upper bound of  $b(n, p)$ .

### Preventing cycles of fixed length.

In a different direction, let  $\mathcal{C}_k$  be the property of a tournament to contain an induced copy of a (directed) cycle of length  $k$ . Note that the properties  $\mathcal{C}_3$  and  $\mathcal{C}$  are equivalent, since if a tournament contains a cycle (of some length  $k$ ), then it contains a cycle of length three. Therefore, preventing a cycle of length  $k$  when playing on  $K_n$  is at least as easy for OBreaker as preventing a cycle of length three. Or in other words,  $t(n, \mathcal{C}_k) \leq t(n, \mathcal{C})$  for all  $3 \leq k \leq n$ . Note that the property  $\mathcal{C}_k$  does not necessarily imply the property  $\mathcal{C}_{k-1}$ , so the sequence  $t(n, \mathcal{C}_3), t(n, \mathcal{C}_4), \dots$  is not necessarily monotone. The threshold bias for the Hamilton cycle game is  $t(n, \mathcal{C}_n) = (1 + o(1))n/\ln n$ , as was recently proved in [6]. We wonder how  $t(n, \mathcal{C}_k)$  behaves as a function of  $n$  and  $k$ . In particular, it would be interesting to know for which values of  $k = k(n)$  besides  $n$  the function  $t(n, \mathcal{C}_k)$  is sublinear.

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